Contact between representative rough surfaces

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Abstract

A numerical analysis of mechanical frictionless contact between rough self-affine elastic manifolds was carried out. It is shown that the lower cutoff wavenumber in surface spectra is a key parameter controlling the representativity of the numerical model. Using this notion we demonstrate that for representative surfaces the evolution of the real contact area with load is universal and independent of the Hurst roughness exponent. By introducing a universal law containing three constants, we extend the study of this evolution beyond the limit of infinitesimal area fractions.

Keywords: rough contact, elastic contact, roughness, true contact area

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Contents

1 Introduction

Real surfaces are self-affine [Meakin, 1998, Krim and Palasantzas, 1995] and the surface heights are distributed normally [Greenwood and Williamson, 1966]. Due to this roughness the real contact area A is often only a small fraction of the nominal or apparent contact area A_0. The real contact area fraction A/A_0 and its evolution determines the contact resistivity, friction and the transfer of energy (heat and electric charge) through the contact interface. Thus understanding the contact-area evolution has profound implications in various fundamental (e.g., origin of friction) and engineering studies (e.g., electro-mechanical contact, tire-road interaction). From early experiments [Bowden and Tabor, 2001] and analytical theories [Greenwood and Williamson, 1966, Bush et al., 1975], it has been considered as an established knowledge that A evolves linearly with the normal load F at relatively small fractions of contact. It was confirmed by numerical simulations of normal frictionless elastic contact between rough surfaces [Hyun et al., 2004], which made considerable progress since then. Nowadays, the most advanced computations exploit fine surface discretizations (2048×2048 [Campañá and Müser, 2007, Campañá et al., 2011, Almqvist et al., 2011, Pohrt and Popov, 2012], 4096×4096 [Campañá et al., 2008]) to extract statistically meaningful results valid even for small fractions of contact. Many of the numerical investigations are based on synthesized surfaces preserving the aspect of self-affinity. In early studies [Hyun et al., 2004, Pei et al., 2005] the artificial rough surfaces obeyed fractality which scaled down to the domain discretization [Fournier et al., 1982]. Later it was realized that the surface has to be smooth enough [Campañá and Müser, 2007, Yastrebov et al., 2011] to represent correctly the mechanics of contact and to obtain a reliable estimation of the contact area growth comparable with analytical theories. This smoothness implies a sharp decay of the power spectrum density for wavelengths smaller than short cutoff wavelength λ_s. On the other hand, to the best of our knowledge, there are no consistent study on the influence of long cutoff wavelength λ_l on the response of rough surfaces. Despite remarks made in [Hyun and Robbins, 2007] on the importance of λ_l for mechanics of rough contact, in almost all investigations of the real-contact-area evolution [Hyun et al., 2004, Campañá and Müser, 2007, Pohrt and Popov, 2012, Putignano et al., 2012] λ_l was limited by the size of the specimen $\lambda_l = L$. In this article we show that λ_l plays a crucial role in numerically precise theory of rough contact, because it controls the representativity of the simulated system.

In the limit of infinitesimal contact, analytical theories predict that the real contact area A evolves proportionally to the normal force F with coefficient κ normalized by the root mean squared slope of the surface $\sqrt{\langle|\nabla h|^2\rangle}$ and the effective Young's modulus E^* [Johnson, 1987], which gives

$$A = \kappa / \sqrt{\langle|\nabla h|^2\rangle}\, F/E^*. \quad (1)$$

According to asperity-based [Greenwood and Williamson, 1966] models (Bush, Gibson, Thomas (BGT) [Bush et al., 1975], Greenwood [Greenwood, 2006],

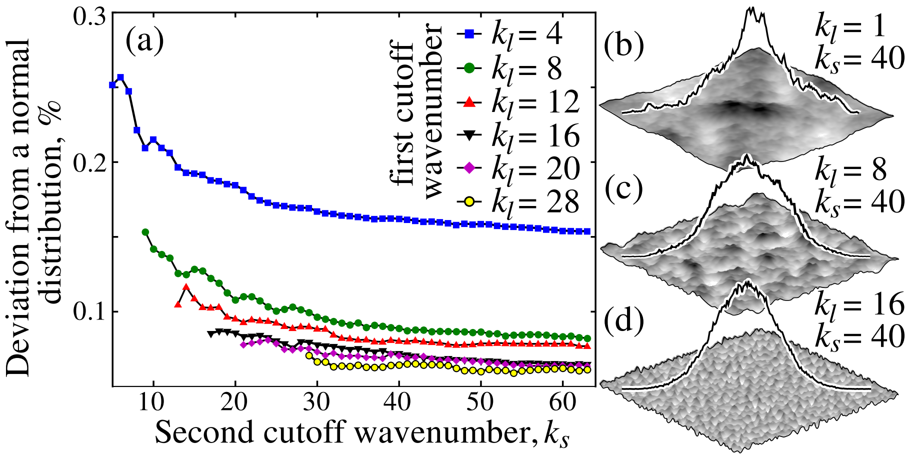

Figure 1: Representativity of a rough self-affine surface can be estimated according to the normality of heights distribution. Here, the criterion of proximity is an error between the distribution of surface's heights and the associated normal distribution; (a) this error is strongly dependent on the cutoff wavenumbers k_s and k_l. Every point on the graph represents an error averaged over 30 statistically equivalent surfaces. Examples of rough surfaces and the corresponding heights distributions for different cutoffs $k_s = 40$ and (b) $k_l = 1$, (c) $k_l = 4$, (d) $k_l = 16$ are also depicted.

Nayak-Thomas [Thomas, 1999]) $\kappa = \sqrt{2\pi}$ but Eq. (1) is valid only in an asymptotic limit for infinitesimal fractions of the contact area $A/A_0 \to 0$. The convergence of κ with decreasing fraction A/A_0 is slow and depends on the bandwidth parameter α introduced in [Longuet-Higgins, 1957] as $\alpha = m_0 m_4/m_2^2$, where $m_i = \int_{k_l}^{k_s} k^i C(k) dk$ is the i-th moment of the power spectrum density $C(k)$ of the surface. For instance, for $\alpha = 10$, κ is overestimated by about 12% for $A/A_0 = 10^{-5}$ and this error reduces only to 4% when $A/A_0 = 10^{-10}$ [Greenwood, 2006]. Consequently, using a complete numerical model it is not possible to demonstrate the linearity predicted by Eq. (1) nor to approach its asymptotic limit. It is also worth noting that in forementioned models the evolution of the real contact area is strictly nonlinear for realistic fractions of contact [Carbone and Bottiglione, 2008, Paggi and Ciavarella, 2010]. Moreover, this evolution does not depend on the Hurst roughness exponent H but only on α [Carbone and Bottiglione, 2008]. A competing theory was proposed by Persson [Persson, 2001b, Persson, 2001a], it also predicts a linear contact area evolution for small contact fractions, however, the obtained coefficient of proportionality $\kappa = \sqrt{8/\pi}$ is significantly smaller than in the asperity-based models. As in the latter models, the evolution of the contact area does not depend on the Hurst exponent. Numerical studies [Hyun et al., 2004, Campañá and Müser, 2007, Pohrt and Popov, 2012, Putignano et al., 2012] demonstrated an approximately linear evolution of the contact area with load. But in contrast to the analytical models, κ was shown to depend on the Hurst exponent and to be confined between the asymptotic limits of the BGT [Bush et al., 1975] and Persson [Persson, 2001b, Persson, 2001a] theories. In this article, by introducing the notion of surface *representativity*, we obtain qualitatively new numerical results, which we believe correctly represent the mechanical response of realistic rough surfaces. Moreover, they appeal to a broad interpretation of the data not restricted to a single proportionality coefficient between contact area and load for infinitesimal contact fractions.

2 Properties of generated surfaces

2.1 Representativity of surfaces

A meaningful numerical simulation of rough contact has to be carried out on a representative self-affine surface element (RSSE) (e.g. [Kanit et al., 2003]) either generated numerically or chosen from experimental measurements. The reason to use generated rough surfaces is that they may be obtained for any cutoff wavelengths and Hurst exponent H. On the one hand, the RSSE has to be large enough to obtain a similar response for different realizations of statistically equivalent surfaces. On the other hand, it has to be as small as possible to retain a numerically solvable contact problem. So the RSSE is defined according to the permitted error between mechanical responses of surfaces of different sizes, e.g. L and $2L$. This mechanical representativity can be linked to the geometrical representativity, which we define as the proximity of the surface heights distribution to a normal distribution [Yastrebov et al., 2011]. In particular, this proximity is important in the range of maximal heights corresponding to the only zones which come in contact at small loads. To analyze how the representativity depends on the cutoffs, for each pair of cutoff wavenumbers $k_s = L/\lambda_s$ and $k_l = L/\lambda_l$ we generate 30 statistically equivalent surfaces using FFT filtering algorithm [Hu and Tonder, 1992]. The power spectra of generated surfaces follow accurately power law $C(k) \sim k^{-2(H+1)}$ with given Hurst exponent H. The size of surfaces is $L = 1$, the discretization spacing is $\Delta L = 1/1024$ and $H = 0.8$. Fig. 1 represents the average $L2$ error between the heights distribution function of the generated surfaces and the corresponding normal distributions $G(h, \bar{h}, \sigma)$ for different cutoffs

$$\varepsilon(h, N) = \left[\int\limits_{-\infty}^{+\infty} \Big(G(h, \bar{h}, \sigma) - P_N(h) \Big)^2 \, dh \right]^{1/2}$$

where the mean height $\bar{h}$, the standard deviation $\sigma = \sqrt{\langle (\bar{h} - h)^2 \rangle}$, and the probability density function (PDF) of heights $P_N(h)$ are extracted from the generated surface and N is the number of bins used to evaluate the PDF of the surface. Function G stands for a normal distribution. The error depicted in Fig. 1 is computed for $N = 500$. It follows from this figure that a generated surface becomes more and more representative, when the first cutoff

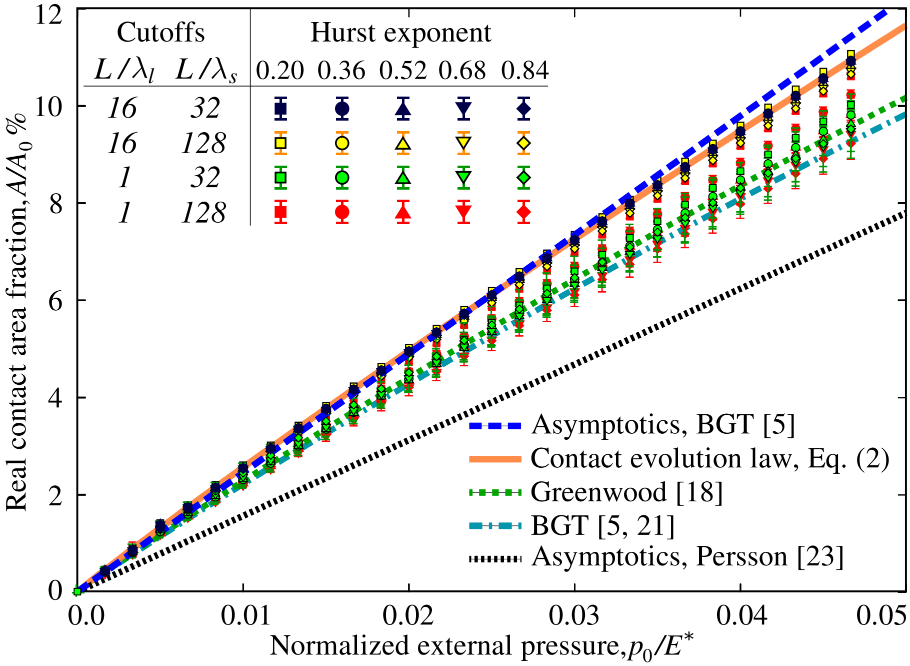

Figure 2: Evolution of the real contact area fraction A/A_0 with normalized external load p_0/E^* are plotted for different cutoff wavelengths and Hurst exponents; the rms slope is kept constant for all surfaces $\sqrt{\langle|\nabla h|^2\rangle} = 0.1$. The results for representative surfaces ($\lambda_l = L/16$) are in a good agreement with the asymptotic limit of the Bush, Gibson, Thomas (BGT) theory [Bush et al., 1975] and are perfectly fitted by the suggested contact evolution law Eq. (2) ($\kappa = 1.145\sqrt{2\pi}, \mu = 0.55, \beta = 0.21$). Moreover, if the surface is sufficiently smooth $\lambda_s = L/32$ we do not observe any dependence of results on the Hurst exponent. The evolution of the real contact area predicted by Greenwood [Greenwood, 2006] and BGT [Bush et al., 1975] theories for spectrum bandwidth $m_0m_4/m_2^2 = 2$ (data from [Carbone and Bottiglione, 2008]) lie in the confidence interval of the results obtained for non-representative surfaces ($\lambda_l = L$) with high Hurst exponents $H \gtrsim 0.5$. Asymptotic limit of Persson theory [Persson, 2001b, Persson, 2001a] lies below all obtained results.

wavenumber k_l increases up to $8-16$, whereas the variation of k_s has a weaker effect on the surface representativity if the spectrum is sufficiently rich $k_s \gg k_l$. The choice $\lambda_l = L$ leads to a surface with a mechanical response which varies considerably from one surface realization to another. The fact that in average this response does not correspond to the response of a bigger rough surface with similar random properties is of crucial importance for the study of rough contact. The key reasons for this discrepancy are the long-range interactions ($1/r$) between contacting asperities and inevitable boundary conditions on lateral sides of the specimen (periodic, symmetric, free). Accordingly, the response of any heterogeneous system with random properties and long-range interactions should be studied on a representative system element.

2.2 Smoothness of surfaces

Besides the long cutoff wavelength, in generated or experimental surfaces there is a limitation connected with their inevitable discreteness, either due to a numerical resolution scheme and/or experimental measurements. In real self-affine surfaces the power spectrum density decays as a power-law of the wavenumber k, $C(k) \sim k^{-2(1+H)}$. This law may be preserved down to wavelengths comparable to atomic spacings [Krim and Palasantzas, 1995], where the continuum contact mechanics is not valid anymore [Luan and Robbins, 2005]. To remain in the continuum framework and to capture accurately the mechanics, we introduce a short cutoff wavelength $\lambda_s > \Delta L$, which ensures the smoothness of surfaces at a certain magnification as, for example, in [Campañá and Müser, 2007]. The inherent cutoff $\lambda_s = \Delta L$ used, for example, in [Hyun et al., 2004, Pei et al., 2005] results in a "one node - one asperity" approach, which does not allow to correctly reproduce the local change of the contact force with separation [Yastrebov et al., 2011] nor to estimate a realistic growth of localized contact zones but only the growth of their number. Considering smooth–rough-surfaces with a "truncated self-affinity" $\lambda_s \gg \Delta L$ eliminates these shortcomings.

3 Description of simulations

Using the algorithm [Hu and Tonder, 1992] we generate 12 statistically equivalent surfaces for each pair of cutoff wavelengths $L/\lambda_l = 1, 2, 8, 16$, $L/\lambda_s = 32, 64, 128, 256, 512$ and different Hurst roughness exponents $H = 0.2$, 0.36, 0.52, 0.68, 0.84. For all the surfaces the constant root mean squared slope $\sqrt{\langle|\nabla h|^2\rangle} = 0.1$ is preserved; as before $L = 1$ and $\Delta L = 1/1024$. It is important to note that, in contrast to geometrical estimations of $\sqrt{\langle|\nabla h|^2\rangle}$ [Hyun et al., 2004], which depend on the surface discretization and thus may underestimate significantly the real value, we evaluate the rms slope according to the surface power spectrum [Longuet-Higgins, 1957]. To solve the contact problem between a periodic rigid rough surface and a deformable flat half-space we use the spectral based boundary element method [Stanley and Kato, 1997]

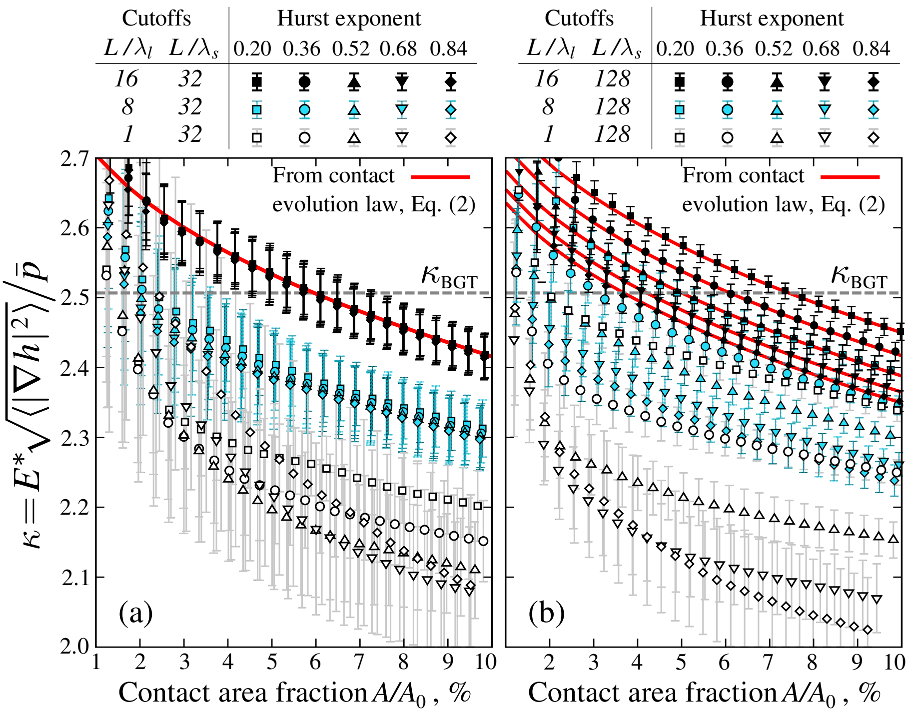

Figure 3: Evolution of κ with the real contact area fraction A/A_0 is depicted for different Hurst roughness exponents and cutoff wavelengths $\lambda_l = L, L/8, L/16$, a) $\lambda_s = L/32$, b) $\lambda_s = L/128$; all results for representative surfaces $\lambda_l = L/16$ are fitted by the contact evolution law Eq. (2) with $\kappa = 1.145\sqrt{2\pi}$, $\mu = 0.55$, $\beta = 0.21$ for (a) $\lambda_l = L/16$, $\lambda_s = L/32$; with $\kappa = 1.145\sqrt{2\pi}$, $\mu = 0.56$ and $\beta \in [1.120,\ 1.172]$ for (b) $\lambda_l = L/16$, $\lambda_s = L/128$. For comparison, the asymptotic limit $\kappa_{\text{BGT}} = \sqrt{2\pi}$ from Bush, Gibson, Thomas theory [Bush et al., 1975] is plotted, while the asymptotic limit of Persson theory [Persson, 2001b, Persson, 2001a] $\kappa_{\text{P}} = \sqrt{8/\pi} \approx 1.60$ is out of the plot range.

with some minor improvements[1]. This method allows to solve accurately the equations of continuum mechanics under contact constraints and, in contrast to asperity-based models, takes into account all underlying mechanics: complex shapes of asperities, junction of contact zones associated with different asperities and long-range deformation of the elastic half-space in response to contact forces. For each simulation the contact area fraction reaches $\approx 10\%$ under the external pressure linearly increasing up to $p_0/E^* = 0.05$ within 30 increments. In Fig. 2 we compare the area-force curves of our numerical results and several analytical theories, which were computed and summarized in [Carbone and Bottiglione, 2008]. To demonstrate better the nonlinearity of the area evolution for surfaces with different cutoffs and Hurst exponents, we present in Fig. 3 the proportionality coefficient κ expressed from Eq. 1 as $\kappa = E^* \sqrt{\langle|\nabla h|^2\rangle}/\bar{p}$, where $\bar{p} = F/A$ is the mean contact pressure.

4 Results

The value of κ for the observed range (up to 10% of the contact fraction) is a decreasing function of the area. For non-representative $\lambda_l < L/16$ or too rough $\lambda_s < 32\Delta L$ surfaces, there is a clear tendency (see Fig. 3, b): a higher H results in a smaller κ. This dependence of the results on the Hurst exponent corresponds to all up to date numerical investigations of the κ constant [Hyun et al., 2004, Campañá and Müser, 2007, Putignano et al., 2012, Pohrt and Popov, 2012]. However, we argue that these results are strongly affected by non-representativity and an excessive roughness of exploited surfaces. In contrast to these results, if the mechanics of contact is well resolved ($\lambda_s \gg \Delta L$) and the surface is representative ($\lambda_l \gtrapprox L/16$) the real contact area evolution does not depend anymore on the Hurst exponent (see dark symbols in Fig. 3, a). This result is in a good agreement with analytical theories and has never been obtained before in complete numerical models. Reducing λ_l reduces the data scatter, which is one of criteria of mechanical representativity. The data scatter increases with increasing H, thus, the smoother the surface, the bigger RSSE is needed to capture an average mechanical behavior. The rise in κ with decreasing λ_l can be interpreted in terms of Persson theory [Persson, 2001b, Persson, 2001a]: reducing λ_l is equivalent to decreasing magnification which rises κ. Note that our results for κ are not confined between the asymptotic limits of the BGT [Bush et al., 1975] $\kappa_{\text{BGT}} = \sqrt{2\pi}$ and Persson theories $\kappa_{\text{P}} = \sqrt{8/\pi}$ [Persson, 2001b], as was generally observed by other authors [Hyun et al., 2004, Campañá and Müser, 2007, Putignano et al., 2012, Pohrt and Popov, 2012].

[1] The original formulation of the resolution algorithm [Stanley and Kato, 1997] contains some errors. The most important is the shift of the solution in Fourier space by one wavenumber. A corrected version of this method, validated on many cases, was used for the current study.

4.1 Contact evolution law

In light of these results, we propose a new phenomenological contact evolution law which is based on our observations of the change in the mean contact pressure $\bar{p} = F/A$ with respect to the applied pressure $p_0 = F/A_0$ depicted in Fig. 4. For considered interval of contact areas $A/A_0 \in [0.01, 0.11]$ the decay of $\partial\bar{p}/\partial p_0$ can be approximated by power-law $\partial\bar{p}/\partial p_0 = \beta(A_0/A)^\mu$, $0 < \mu < 1$. Therefore, integrating the latter expression, we obtain the evolution of the real contact area as

$$A/A_0 = \left(\beta + \left[\kappa p_0/(\sqrt{\langle|\nabla h|^2\rangle}E^*)\right]^{\mu-1}\right)^{1/(\mu-1)} \tag{2}$$

where κ is still a proportionality constant between the area and normalized force for infinitesimal pressures, two other constants μ and β can be easily and uniquely found from the data analysis. For converged results $\lambda_l = L/16$, $\lambda_s = L/32$ by fitting the normalized inverse mean pressure (Fig. 3) we found $\kappa = (1.145 \pm 0.002)\sqrt{2\pi}$, which is about 15% higher than the asymptotic limit of the BGT theory [Bush et al., 1975]. From Fig. 4 we uniquely obtain $\mu = 0.55 \pm 0.02$, $\beta = 0.210 \pm 0.005$.

5 Conclusions

To study the growth of the real contact area between rough self-affine manifolds we obtained statistically meaningful results for different cutoff wavelengths and Hurst exponents. We demonstrated that the real contact area evolves nonlinearly with applied force even for reasonably small contact fractions $A/A_0 < 0.1$. Note that an almost linear evolution was observed for rough contact between elasto-plastic materials [Persson, 2001a, Pei et al., 2005, Yastrebov et al., 2011], however, in this case the Johnson's assumption [Johnson, 1987] on replacing two deformable solids by one solid with effective elasto-plastic properties and superposed roughnesses is not verified and a full simulation of two deformable solids is required.

In conclusion, we state that to obtain realistic results in rough contact analysis, one needs to construct representative surfaces. This representativity necessarily requires the long wavelength cutoff to be significantly smaller than the specimen size $\lambda_l \gtrapprox L/16$. The Hurst exponent does not change the evolution of the real contact area with load if both the representativity and the smoothness of surfaces are maintained. To describe the evolution of the real contact area we proposed a new phenomenological law which describes well our results for moderate pressures and reduces to the classical BGT law [Bush et al., 1975] in the limit of infinitesimal contact fractions. However, further work is required to verify the universality of this law. In perspective we aim to study the pressure distribution and the evolution of the contact area up to the full contact, for which the numerical results should be in a better agreement with Persson theory [Persson, 2001b, Persson, 2001a]. An important question, which we could not answer in our study, is how the bandwidth parameter influences the mechanical behavior of rough surfaces.

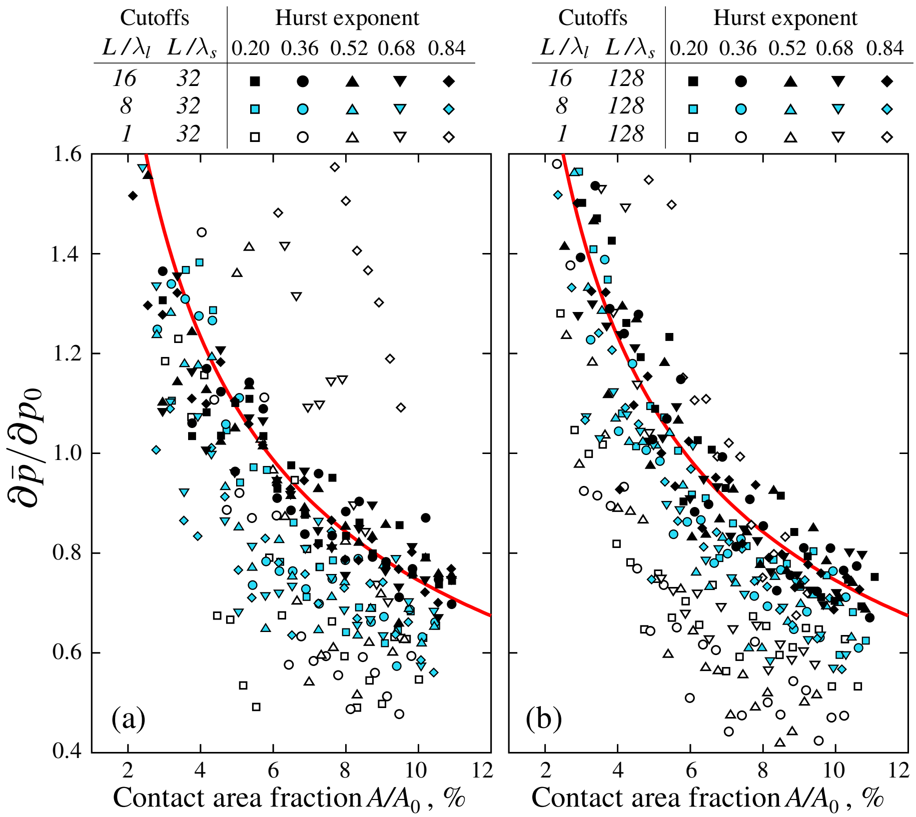

Figure 4: The mean pressure derivative with respect to the external pressure $\partial\bar{p}/\partial p_0$ decreases nonlinearly with increasing fraction of the real contact area A/A_0. We plot its evolution for different Hurst roughness exponents and cutoff wavelengths $\lambda_l = L$, $L/8$, $L/16$ and (a) $\lambda_s = L/32$, (b) $\lambda_s = L/128$. For representative surfaces the results can be roughly approximated by a power law $\partial\bar{p}/\partial p_0 = \beta(A_0/A)^{\mu}$; solid line corresponds to parameters $\beta = 0.21$, $\mu = 0.55$, which fit well our results for representative $\lambda_l = L/16$ smooth surfaces.

To address it, one needs to carry out numerical simulations on a considerably finer discretization of surfaces than was reported here. We hope that our results will motivate new simulations based on the notion of representativity and taking into account friction, visco-plasticity, adhesion and surface energy.

6 Acknowledgments

The financial support from the European Research Council (ERCstg UFO-240332) is greatly acknowledged.

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