

# Contact mechanics and elements of tribology

## Lecture 5.

### *Computational Contact Mechanics*

Vladislav A. Yastrebov

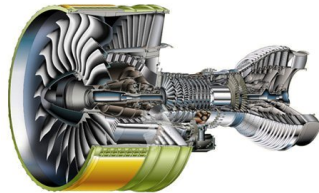
*MINES ParisTech, PSL Research University, Centre des Matériaux, CNRS UMR 7633,  
Evry, France*

@ Centre des Matériaux  
January 25, 2016

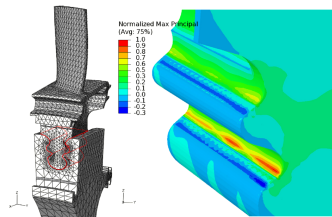
- Introduction
- Governing equations
- Optimization methods
- Resolution algorithm
- Examples

# Industrial and natural contact problems

## 1 Assembled parts, e.g. engines



Aircraft's engine GP 7200  
[www.safran-group.com](http://www.safran-group.com)



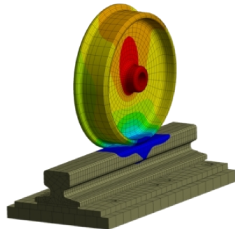
[1] M. W. R. Savage  
*J. Eng. Gas Turb. Power*, 134:012501 (2012)

# Industrial and natural contact problems

- 1 Assembled parts, e.g. engines
- 2 Railroad contacts



High speed train TGV [www.sncf.com](http://www.sncf.com)



Wilde/ANSYS [wildeanalysis.co.uk](http://wildeanalysis.co.uk)



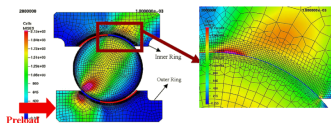
# Industrial and natural contact problems

- 1 Assembled parts, e.g. engines
- 2 Railroad contacts
- 3 Gears and bearings



Bearings

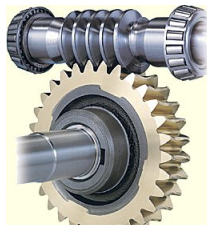
[www.skf.com](http://www.skf.com)



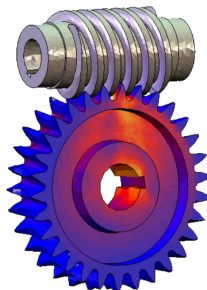
[1] F. Massi, J. Rocchi, A. Culla, Y. Berthier  
*Mech. Syst. Signal Pr.*, 24:1068-1080 (2010)

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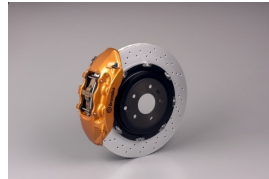
Helical gear [www.tpg.com.tw](http://www.tpg.com.tw)



[www.mscsoftware.com](http://www.mscsoftware.com)

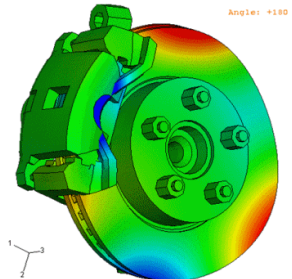
# Industrial and natural contact problems

- 1 Assembled parts, e.g. engines
- 2 Railroad contacts
- 3 Gears and bearings
- 4 Breaking systems



*Assembled breaking system*

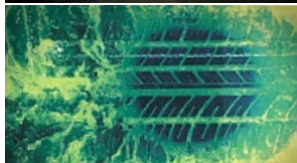
[www.brembo.com](http://www.brembo.com)



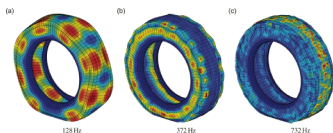
[www.mechanicalengineeringblog.com](http://www.mechanicalengineeringblog.com)

# Industrial and natural contact problems

- 1 Assembled parts, e.g. engines
- 2 Railroad contacts
- 3 Gears and bearings
- 4 Breaking systems
- 5 Tire-road contact



Tire-road contact [www.michelin.com](http://www.michelin.com)



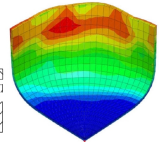
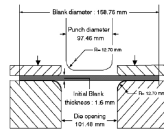
[1] M. Brinkmeier, U. Nackenhorst, S. Petersen, O. von Estorff, *J. Sound Vib.*, 309:20-39 (2008)

# Industrial and natural contact problems

- 1 Assembled parts, e.g. engines
- 2 Railroad contacts
- 3 Gears and bearings
- 4 Breaking systems
- 5 Tire-road contact
- 6 Metal forming



Deep drawing [www.thomasnet.com](http://www.thomasnet.com)



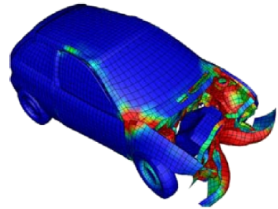
[1] G. Rousselier, F. Barlat, J. W. Yoon  
*Int. J. Plasticity*, 25:2383-2409 (2009)

# Industrial and natural contact problems

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- 3 Gears and bearings
- 4 Breaking systems
- 5 Tire-road contact
- 6 Metal forming
- 7 Crash tests



Crash-test [www.porsche.com](http://www.porsche.com)



[1] O. Klyavin, A. Michailov, A. Borovkov  
[www.fea.ru](http://www.fea.ru)

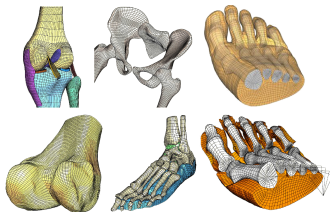
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- 7 Crash tests
- 8 Biomechanics



Human articulations

[www.sportssupplements.net](http://www.sportssupplements.net)



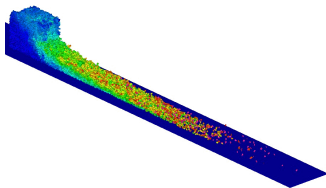
J. A. Weiss, University of Utah  
Musculoskeletal Research Laboratories

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- 9 Granular materials



Sand dunes [www.en.wikipedia.org](http://www.en.wikipedia.org)



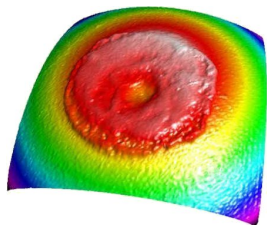
Azema et al, LMGC90

E.

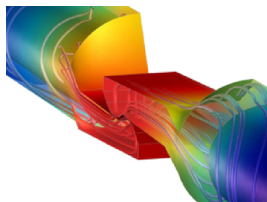


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- 6 Metal forming
- 7 Crash tests
- 8 Biomechanics
- 9 Granular materials
- 10 Electric contacts



*Damage at electric contact zone*  
[www.taicaan.com](http://www.taicaan.com)



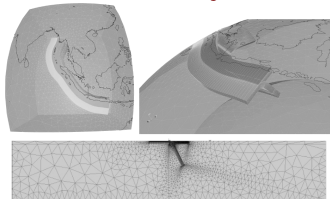
*Simulation of electric current*  
[www.comsol.com](http://www.comsol.com)

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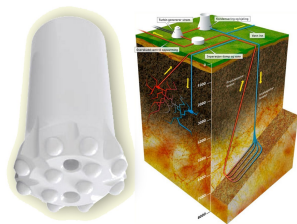
*San-Andreas fault, by M. Rightmire  
[www.sciencedude.ocregister.com](http://www.sciencedude.ocregister.com)*



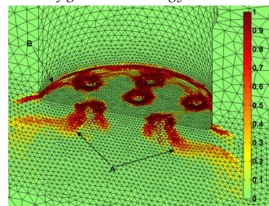
*[1] J.D. Garaud, L. Fleitout, G. Cailletaud  
Colloque CSMA (2009)*

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- 8 Biomechanics
- 9 Granular materials
- 10 Electric contacts
- 11 Tectonic motions
- 12 Deep drilling



Drill Bit tool *RobitRocktools*;  
extraction of geothermal energy (*SINTEF, NTNU*)



[1] T. Saksala, *Int. J. Numer. Anal. Meth. Geomech.* (2012)

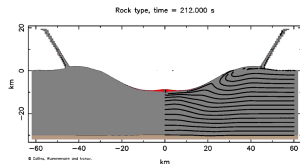
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*Impact crater, Arizona*

[www.MrEclipse.com](http://www.MrEclipse.com) et [maps.google.com](http://maps.google.com)



*Simulation of formation of Copernicus crater*  
Yue Z., Johnson B. C., et al. *Projectile*

remnants in central peaks of lunar impact  
craters. *Nature Geo* 6 (2013)

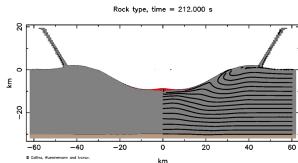
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- 14 etc.



Impact crater, Arizona

[www.MrEclipse.com](http://www.MrEclipse.com) et [maps.google.com](http://maps.google.com)



Simulation of formation of Copernicus crater  
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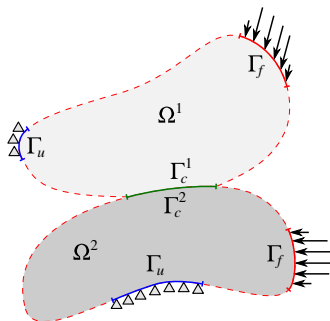
# Equilibrium and contact conditions

## ■ Balance of momentum

$$\begin{cases} \nabla \cdot \underline{\underline{\sigma}} + \underline{\underline{f}}_v = 0 & \text{in } \Omega_{1,2} \\ \underline{\underline{\sigma}} \cdot \underline{\underline{n}} = \underline{\underline{t}}_0 & \text{on } \Gamma_f \\ \underline{\underline{u}} = \underline{\underline{u}}_0 & \text{on } \Gamma_u \\ ? & \text{on } \Gamma_c \end{cases}$$

## ■ Frictionless contact conditions (*intuitive*)

- 1 No penetration
- 2 No adhesion
- 3 No shear transfer



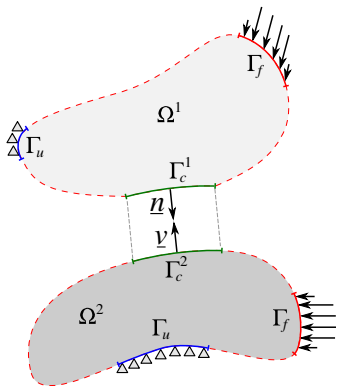
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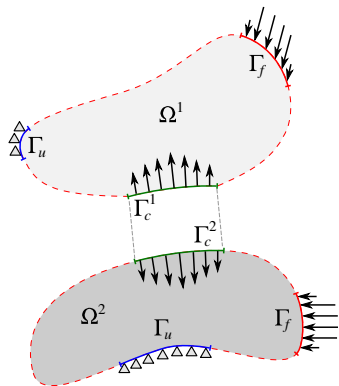
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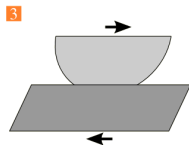
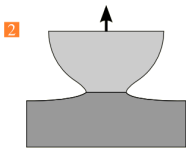
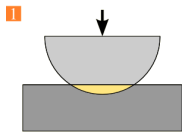
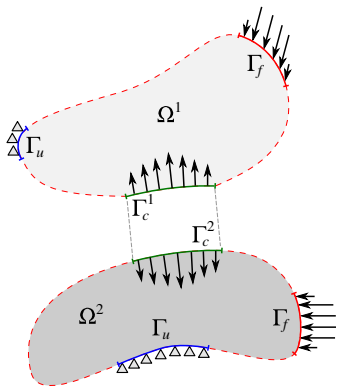
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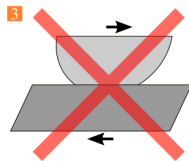
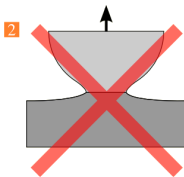
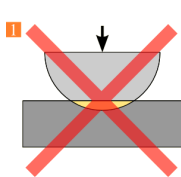
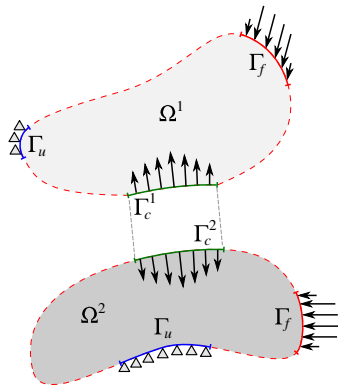
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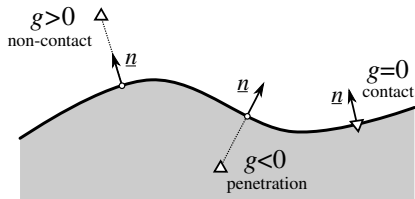
# Gap function

## ■ Gap function $g$

- gap = - penetration
- asymmetric function
- defined for
  - separation  $g > 0$
  - contact  $g = 0$
  - penetration  $g < 0$
- governs normal contact

## ■ Master and slave split

*Gap function is determined for all slave points with respect to the master surface*



*Gap between a slave point and a master surface*

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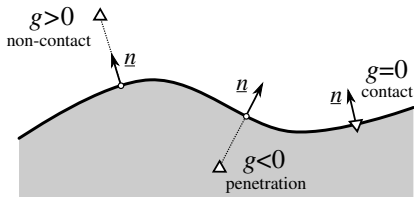
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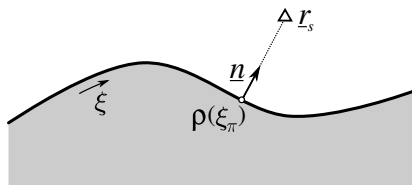
## ■ Normal gap

$$g_n = \underline{n} \cdot [\underline{r}_s - \underline{\rho}(\xi_\pi)]$$

$\underline{n}$  is a unit normal vector,  $\underline{r}_s$  slave point,  $\underline{\rho}(\xi_\pi)$  projection point at master surface



*Gap between a slave point and a master surface*



*Definition of the normal gap*

Consider existence and uniqueness



# Frictionless or normal contact conditions

## ■ No penetration

*Always non-negative gap*

$$g \geq 0$$

## ■ No adhesion

*Always non-positive contact pressure*

$$\sigma_n^* \leq 0$$

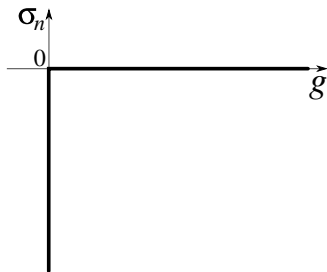
## ■ Complementary condition

*Either zero gap and non-zero pressure, or non-zero gap and zero pressure*

$$g \sigma_n = 0$$

## ■ No shear transfer (automatically)

$$\underline{\sigma}_t^{**} = 0$$



Scheme explaining normal contact conditions

---

$$\sigma_n^* = (\underline{\underline{\sigma}} \cdot \underline{\underline{n}}) \cdot \underline{\underline{n}} = \underline{\underline{\sigma}} : (\underline{\underline{n}} \otimes \underline{\underline{n}})$$

$$\underline{\underline{\sigma}}_t^{**} = \underline{\underline{\sigma}} \cdot \underline{\underline{n}} - \sigma_n \underline{\underline{n}} = \underline{\underline{n}} \cdot \underline{\underline{\sigma}} \cdot (\underline{\underline{I}} - \underline{\underline{n}} \otimes \underline{\underline{n}})$$

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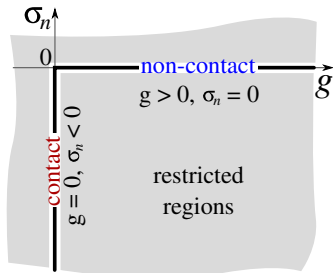
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**Improved scheme explaining normal contact conditions**

# Frictionless or normal contact conditions

In mechanics:

*Normal contact conditions*

$\equiv$

*Frictionless contact conditions*

$\equiv$

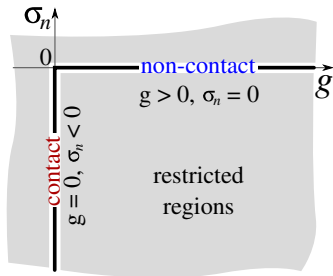
*Hertz<sup>1</sup>-Signorini<sup>[2]</sup> conditions*

$\equiv$

*Hertz<sup>1</sup>-Signorini<sup>[2]</sup>-Moreau<sup>[3]</sup> conditions*

also known in **optimization theory** as

*Karush<sup>[4]</sup>-Kuhn<sup>[5]</sup>-Tucker<sup>[6]</sup> conditions*



**Improved scheme explaining normal contact conditions**

$$g \geq 0, \quad \sigma_n \leq 0, \quad g\sigma_n = 0$$

<sup>1</sup>Heinrich Rudolf Hertz (1857–1894) a German physicist who first formulated and solved the frictionless contact problem between elastic ellipsoidal bodies.

<sup>2</sup>Antonio Signorini (1888–1963) an Italian mathematical physicist who gave a general and rigorous mathematical formulation of contact constraints.

<sup>3</sup>Jean Jacques Moreau (1923) a French mathematician who formulated a non-convex optimization problem based on these conditions and introduced pseudo-potentials in contact mechanics.

<sup>4</sup>William Karush (1917–1997), <sup>5</sup>Harold William Kuhn (1925) American mathematicians,

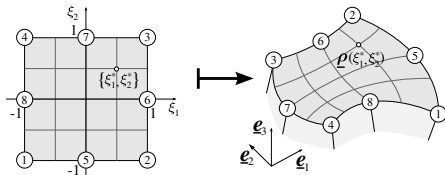
<sup>6</sup>Albert William Tucker (1905–1995) a Canadian mathematician.

# Relative sliding

## Recall:

- Convective coordinate in parent space  $\xi_i \in (-1; 1)$
- Mapping to real space

$$\underline{\rho}(\xi_1, \xi_2, t) = \sum_{i=1}^8 N^i(\xi_1, \xi_2) \underline{\rho}^i(t)$$



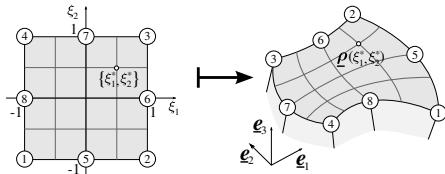


# Relative sliding

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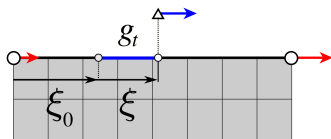
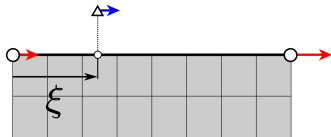
## ■ Tangential slip velocity $\underline{v}_t$

must take into account:

- only tangential component
- relative rigid body motion
- master's deformation

$$\underline{v}_t = \frac{\partial \underline{\rho}}{\partial \xi_1} \dot{\xi}_1 + \frac{\partial \underline{\rho}}{\partial \xi_2} \dot{\xi}_2$$

where  $\partial \underline{\rho} / \partial \xi_i$  are the tangent vectors of the local basis and  $\xi_i$  are the convective coordinates.



*Relative slip between a slave point and a deformable master surface*

# Relative sliding: example

Consider a one-dimensional example:

$P$  is a projection of  $A$  on segment  $BC$ .

$$x_P = \xi x_C + (1 - \xi)x_B \quad (1)$$

Velocity of the projection point

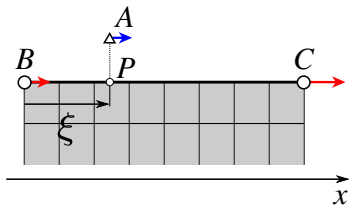
$$\dot{x}_P = \underbrace{\xi \dot{x}_C + (1 - \xi) \dot{x}_B}_{\frac{\partial x_P}{\partial t}} + \underbrace{(x_C - x_B) \dot{\xi}}_{\frac{\partial x_P}{\partial \xi} \dot{\xi}}$$

Subtract the velocity of point  $x_P$  for fixed  $\xi$

$$v_t = \dot{x}_P - \frac{\partial x_P}{\partial t} = (x_C - x_B) \dot{\xi} = \frac{\partial x}{\partial \xi} \dot{\xi}$$

Compute tangential slip increment

$$\Delta g_t^{n+1} = \left. \frac{\partial x}{\partial \xi} \right|_{\xi^n} (\xi^{n+1} - \xi^n)$$



Example of a one-dimensional relative slip

# Relative sliding: example

Consider a one-dimensional example:

$P$  is a projection of  $A$  on segment  $BC$ .

$$x_P = \xi x_C + (1 - \xi)x_B \quad (1)$$

Velocity of the projection point

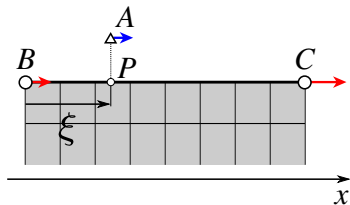
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Ship-river analogy

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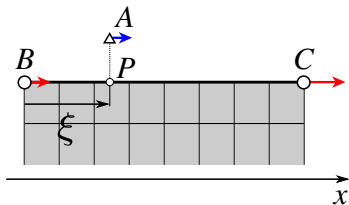
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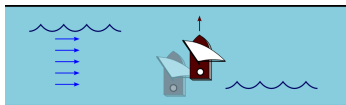
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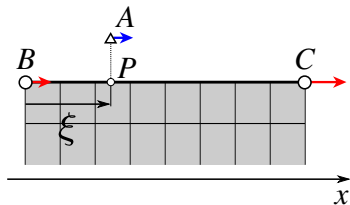
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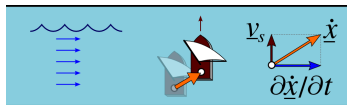
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Example of a one-dimensional relative slip



Ship-river analogy

**Li derivative:** the change of a vector field along the change of another vector field

# Amontons-Coulomb's friction

- **No contact**  $g > 0, \sigma_n = 0$

- **Stick**  $|\underline{v}_t| = 0$

*Inside slip surface/Coulomb's cone*

$$f = |\underline{\sigma}_t| - \mu|\sigma_n| < 0$$

- **Slip**  $|\underline{v}_t| > 0$

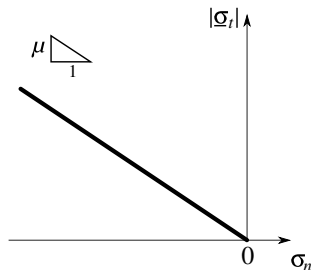
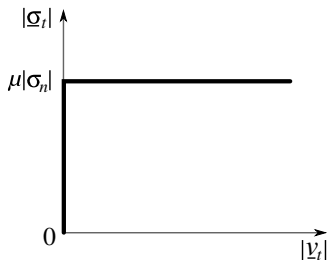
*On slip surface/Coulomb's cone*

$$f = |\underline{\sigma}_t| - \mu|\sigma_n| = 0$$

- **Complementary condition**

*Either zero velocity and negative slip criterion, or non-zero velocity and zero slip criterion*

$$|\underline{v}_t| \left( |\underline{\sigma}_t| - \mu|\sigma_n| \right) = 0$$



Scheme explaining frictional contact conditions

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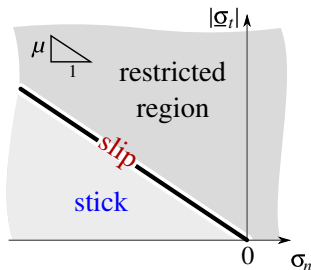
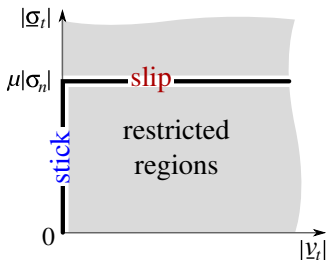
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**Improved** scheme explaining frictional contact conditions

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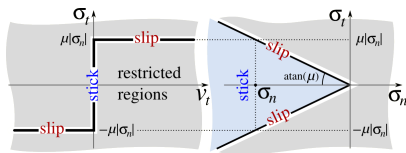
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$$f = |\underline{\sigma}_t| - \mu|\sigma_n| = 0$$

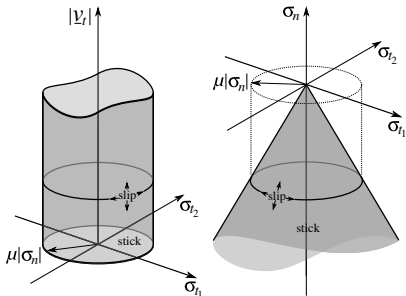
- **Complementary condition**

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$$|\underline{v}_t| \left( |\underline{\sigma}_t| - \mu|\sigma_n| \right) = 0$$



Scheme of 2D frictional contact



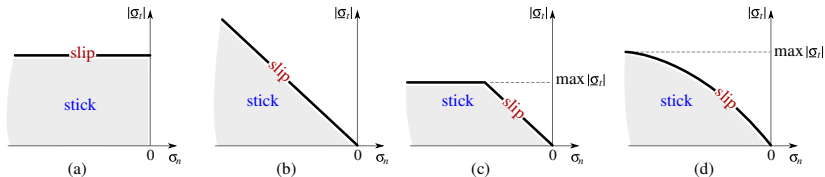
Scheme of 3D frictional contact

$$|\underline{v}_t| \geq 0, \quad |\underline{\sigma}_t| - \mu|\sigma_n| \leq 0, \quad |\underline{v}_t| \left( |\underline{\sigma}_t| - \mu|\sigma_n| \right) = 0$$



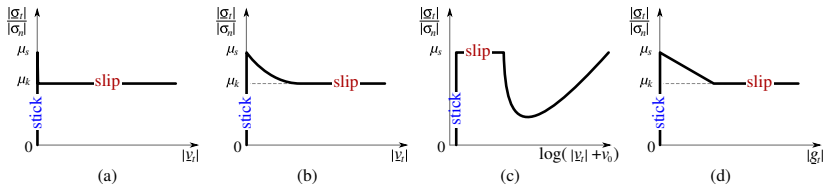
# More friction laws

## • Static criteria



(a) Tresca (b) Amontons-Coulomb (c) Coulomb-Orowan (d) Shaw

## • Kinetic criteria



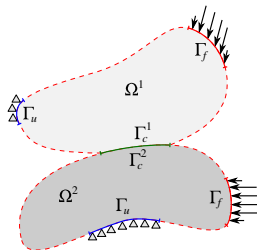
(a,b) velocity weakening (c) velocity weakening-strengthening  
(d) Linear slip weakening

- $\mu_s$  static and  $\mu_k$  kinetic coefficients of friction.

# From strong to weak form

- Balance of momentum and boundary conditions

$$\nabla \cdot \underline{\underline{\sigma}} + \underline{\underline{f}}_v = 0 \quad \text{in } \Omega = \Omega_1 \cup \Omega_2 \quad + \text{B.C.}$$



*Two solids in contact*

# From strong to weak form

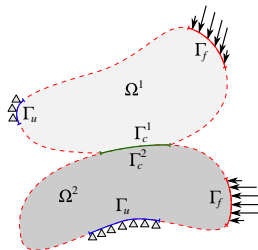
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$$\nabla \cdot \underline{\underline{\sigma}} + \underline{\underline{f}}_v = 0 \quad \text{in } \Omega = \Omega_1 \cup \Omega_2 \quad + \text{B.C.}$$

- Balance of virtual works



$$\boxed{\int_{\partial\Omega} \underline{\underline{n}} \cdot \underline{\underline{\sigma}} \cdot \delta \underline{\underline{u}} d\Gamma} + \int_{\Omega} [\underline{\underline{f}}_v \cdot \delta \underline{\underline{u}} - \underline{\underline{\sigma}} \cdot \delta \nabla \underline{\underline{u}}] d\Omega = 0$$



*Two solids in contact*

# From strong to weak form

- Balance of momentum and boundary conditions

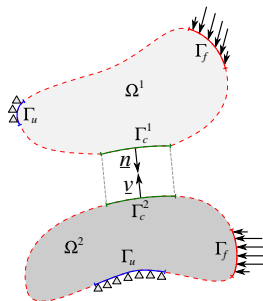
$$\nabla \cdot \underline{\underline{\sigma}} + \underline{\underline{f}}_v = 0 \quad \text{in } \Omega = \Omega_1 \cup \Omega_2 \quad + \text{B.C.}$$

- Balance of virtual works



$$\int_{\partial\Omega} \underline{\underline{n}} \cdot \underline{\underline{\sigma}} \cdot \delta \underline{\underline{u}} d\Gamma =$$

$$\int_{\bar{\Gamma}_1^l} \underline{\underline{n}} \cdot \underline{\underline{\sigma}} \cdot \delta \underline{\underline{\rho}} d\bar{\Gamma}_c^1 + \int_{\bar{\Gamma}_2^r} \underline{\underline{v}} \cdot \underline{\underline{\sigma}} \cdot \delta \underline{\underline{r}} d\bar{\Gamma}_c^2 + \int_{\Gamma_f} \underline{\underline{\sigma}}_0 \cdot \delta \underline{\underline{u}} d\Gamma_f$$



Two solids in contact

# From strong to weak form

- Balance of momentum and boundary conditions

$$\nabla \cdot \underline{\underline{\sigma}} + \underline{\underline{f}}_v = 0 \quad \text{in } \Omega = \Omega_1 \cup \Omega_2 \quad + \text{B.C.}$$

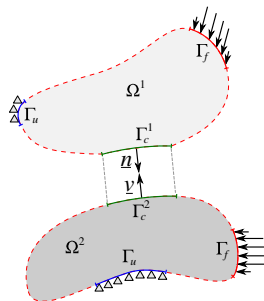
- Balance of virtual works



$$\int_{\partial\Omega} \underline{\underline{n}} \cdot \underline{\underline{\sigma}} \cdot \delta \underline{\underline{u}} d\Gamma \Rightarrow$$

$$\int_{\bar{\Gamma}_1^l} \underline{\underline{n}} \cdot \underline{\underline{\sigma}} \cdot \delta \underline{\underline{\rho}} d\bar{\Gamma}_c^1 + \int_{\bar{\Gamma}_2^l} \underline{\underline{v}} \cdot \underline{\underline{\sigma}} \cdot \delta \underline{\underline{r}} d\bar{\Gamma}_c^2 =$$

$$= \int_{\bar{\Gamma}_1^l} \underline{\underline{n}} \cdot \underline{\underline{\sigma}} \cdot \delta(\underline{\underline{\rho}} - \underline{\underline{r}}) d\bar{\Gamma}_c^1 = \int_{\bar{\Gamma}_1^l} \left( \sigma_n \delta g_n + \underline{\underline{\sigma}}_t^T \delta \underline{\underline{\xi}} \right) d\bar{\Gamma}_c^1$$



Two solids in contact

# From strong to weak form

- Balance of momentum and boundary conditions

$$\nabla \cdot \underline{\underline{\sigma}} + \underline{\underline{f}}_v = 0 \quad \text{in } \Omega = \Omega_1 \cup \Omega_2 \quad + \text{B.C.}$$

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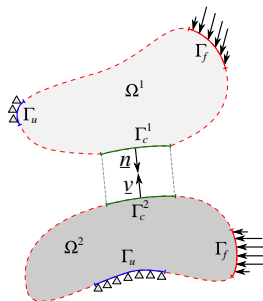


$$\int_{\partial\Omega} \underline{\underline{n}} \cdot \underline{\underline{\sigma}} \cdot \delta \underline{\underline{u}} d\Gamma \Rightarrow$$

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$$\int_{\Omega} \underline{\underline{\sigma}} \cdot \delta \nabla \underline{\underline{u}} d\Omega + \underbrace{\int_{\bar{\Gamma}_1^t} \left( \sigma_n \delta g_n + \underline{\underline{\sigma}}_t^T \delta \underline{\underline{\xi}} \right) d\bar{\Gamma}_c^1}_{\text{Contact term}} = \int_{\Gamma_f} \underline{\underline{\sigma}}_0 \cdot \delta \underline{\underline{u}} d\Gamma + \int_{\Omega} \underline{\underline{f}}_v \cdot \delta \underline{\underline{u}} d\Omega$$



Two solids in contact

# From strong to weak form

- Balance of momentum and boundary conditions

$$\nabla \cdot \underline{\underline{\sigma}} + \underline{\underline{f}}_v = 0 \quad \text{in } \Omega = \Omega_1 \cup \Omega_2 \quad + \text{B.C.}$$

- Balance of virtual works

$$\underbrace{\int_{\Omega} \underline{\underline{\sigma}} \cdot \delta \nabla \underline{\underline{u}} d\Omega}_{\text{Change of the internal energy}} + \underbrace{\int_{\bar{\Gamma}_c^1} \left( \sigma_n \delta g_n + \underline{\underline{\sigma}}_t^T \delta \underline{\underline{\xi}} \right) d\bar{\Gamma}_c^1}_{\text{Contact term}} =$$

Change of the internal energy

Contact term

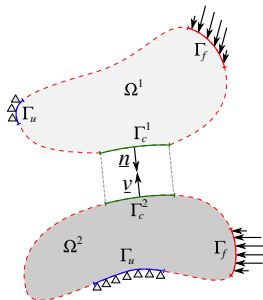
$$\underbrace{\int_{\Gamma_f} \underline{\underline{\sigma}}_0 \cdot \delta \underline{\underline{u}} d\Gamma}_{\text{Virtual work of external forces}} + \underbrace{\int_{\Omega} \underline{\underline{f}}_v \cdot \delta \underline{\underline{u}} d\Omega}_{\text{Virtual work of volume forces}}$$

Virtual work of external forces

Virtual work of volume forces

- Functional space

$\underline{\underline{u}} \in \mathbb{H}^1(\Omega)$  Hilbert space of the first order  
and  $\underline{\underline{u}}$  satisfy boundary and **contact conditions**.



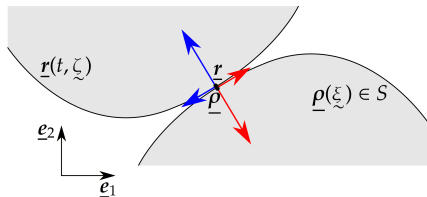
Two solids in contact

# Towards variational inequality

## ■ Contact term

$$\int_{\bar{\Gamma}_c^1} \left( \sigma_n \delta g_n + \underline{\varrho}_i^T \delta \underline{\xi} \right) d\bar{\Gamma}_c^1$$

$$\int_{\bar{\Gamma}_c^1} \sigma_n \delta g_n d\bar{\Gamma}_c^1 \leq 0$$



Contact configuration  $\sigma_n \delta g_n = 0$ ,  $\sigma_n \leq 0$

$$\int_{\Omega} \underline{\underline{\sigma}} \cdot \delta \nabla \underline{u} d\Omega + \int_{\bar{\Gamma}_c^1} \underline{\varrho}_i^T \delta \underline{\xi} d\bar{\Gamma}_c^1 \geq \int_{\Gamma_f} \underline{\sigma}_0 \cdot \delta \underline{u} d\Gamma + \int_{\Omega} \underline{f}_v \cdot \delta \underline{u} d\Omega,$$

$$\mathbb{K} = \left\{ \delta \underline{u} \in \mathbb{H}^1(\Omega) \mid \delta \underline{u} = 0 \text{ on } \Gamma_u, g_n(\underline{u} + \delta \underline{u}) \geq 0 \text{ on } \Gamma_c \right\}$$

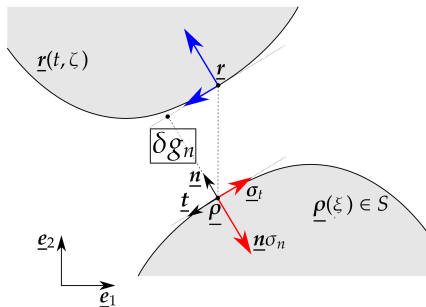


# Towards variational inequality

## ■ Contact term

$$\int_{\bar{\Gamma}_c^1} \left( \sigma_n \delta g_n + \underline{\varrho}_i^T \delta \underline{\xi} \right) d\bar{\Gamma}_c^1$$

$$\int_{\bar{\Gamma}_c^1} \sigma_n \delta g_n d\bar{\Gamma}_c^1 \leq 0$$



*Virtual change of the configuration*

$$\int_{\Omega} \underline{\underline{\sigma}} \cdot \delta \nabla \underline{u} d\Omega + \int_{\bar{\Gamma}_c^1} \underline{\varrho}_i^T \delta \underline{\xi} d\bar{\Gamma}_c^1 \geq \int_{\Gamma_f} \underline{\sigma}_0 \cdot \delta \underline{u} d\Gamma + \int_{\Omega} \underline{f}_v \cdot \delta \underline{u} d\Omega,$$

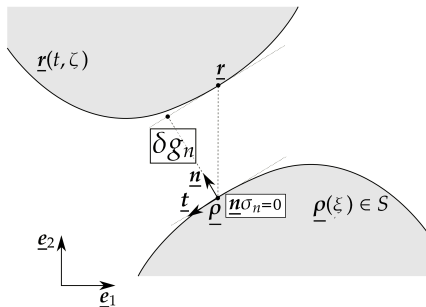
$$\mathbb{K} = \left\{ \delta \underline{u} \in \mathbb{H}^1(\Omega) \mid \delta \underline{u} = 0 \text{ on } \Gamma_u, g_n(\underline{u} + \delta \underline{u}) \geq 0 \text{ on } \Gamma_c \right\}$$

# Towards variational inequality

## ■ Contact term

$$\int_{\bar{\Gamma}_c^1} \left( \sigma_n \delta g_n + \underline{\sigma}_i^T \delta \underline{\xi} \right) d\bar{\Gamma}_c^1$$

$$\int_{\bar{\Gamma}_c^1} \sigma_n \delta g_n d\bar{\Gamma}_c^1 \leq 0$$



Normal term in separation  $\delta g_n > 0$

$$\int_{\Omega} \underline{\underline{\sigma}} \cdot \delta \nabla \underline{u} d\Omega + \int_{\bar{\Gamma}_c^1} \underline{\underline{\sigma}}_i^T \delta \underline{\xi} d\bar{\Gamma}_c^1 \geq \int_{\Gamma_f} \underline{\sigma}_0 \cdot \delta \underline{u} d\Gamma + \int_{\Omega} \underline{f}_v \cdot \delta \underline{u} d\Omega,$$

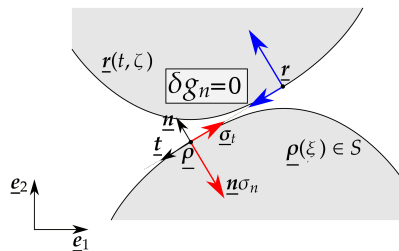
$$\mathbb{K} = \left\{ \delta \underline{u} \in H^1(\Omega) \mid \delta \underline{u} = 0 \text{ on } \Gamma_u, g_n(\underline{u} + \delta \underline{u}) \geq 0 \text{ on } \Gamma_c \right\}$$

# Towards variational inequality

## ■ Contact term

$$\int_{\bar{\Gamma}_c^1} \left( \sigma_n \delta g_n + \underline{\sigma}_i^T \delta \underline{\xi} \right) d\bar{\Gamma}_c^1$$

$$\int_{\bar{\Gamma}_c^1} \sigma_n \delta g_n d\bar{\Gamma}_c^1 \leq 0$$



Normal term in sliding  $\delta g_n = 0$

$$\int_{\Omega} \underline{\underline{\sigma}} \cdot \delta \nabla \underline{u} d\Omega + \int_{\bar{\Gamma}_c^1} \underline{\sigma}_i^T \delta \underline{\xi} d\bar{\Gamma}_c^1 \geq \int_{\Gamma_f} \underline{\sigma}_0 \cdot \delta \underline{u} d\Gamma + \int_{\Omega} \underline{f}_v \cdot \delta \underline{u} d\Omega,$$

$$\mathbb{K} = \left\{ \delta \underline{u} \in \mathbb{H}^1(\Omega) \mid \delta \underline{u} = 0 \text{ on } \Gamma_u, g_n(\underline{u} + \delta \underline{u}) \geq 0 \text{ on } \Gamma_c \right\}$$

# Back to variational equality (unconstrained)

- Constrained minimization problem

$$\int_{\Omega} \underline{\underline{\sigma}} \cdot \delta \nabla \underline{\underline{u}} d\Omega + \int_{\bar{\Gamma}_c^1} \underline{\underline{\sigma}}_i^T \delta \underline{\underline{\xi}} d\bar{\Gamma}_c^1 \geq \int_{\Gamma_f} \underline{\underline{\sigma}}_0 \cdot \delta \underline{\underline{u}} d\Gamma + \int_{\Omega} \underline{\underline{f}}_v \cdot \delta \underline{\underline{u}} d\Omega,$$
$$\mathbb{K} = \left\{ \delta \underline{\underline{u}} \in \mathbb{H}^1(\Omega) \mid \delta \underline{\underline{u}} = 0 \text{ on } \Gamma_u, g_n(\underline{\underline{u}} + \delta \underline{\underline{u}}) \geq 0 \text{ on } \Gamma_c \right\}$$

- Use optimization theory to convert to

$$\int_{\Omega} \underline{\underline{\sigma}} \cdot \delta \nabla \underline{\underline{u}} d\Omega + \int_{\bar{\Gamma}_c^1} \underbrace{C(\sigma_n, \sigma_t, g_n, \underline{\underline{\xi}}, \delta \underline{\underline{u}})}_{\text{Contact term}^*} d\bar{\Gamma}_c^1 = \int_{\Gamma_f} \underline{\underline{\sigma}}_0 \cdot \delta \underline{\underline{u}} d\Gamma + \int_{\Omega} \underline{\underline{f}}_v \cdot \delta \underline{\underline{u}} d\Omega,$$

Unconstrained functional space  $\mathbb{K} = \left\{ \delta \underline{\underline{u}} \in \mathbb{H}^1(\Omega) \mid \delta \underline{\underline{u}} = 0 \text{ on } \Gamma_u \right\}$

Contact term\* is defined on the *potential contact zone*  $\bar{\Gamma}_c^1$ .

# Optimization methods: recall

Functional to be minimized  $F(\mathbf{x})$  under constraint  $g(\mathbf{x}) \geq 0$

- Penalty method
- Lagrange multipliers method
- Augmented Lagrangian method

Functional to be minimized  $F(\mathbf{x})$  under constraint  $g(\mathbf{x}) \geq 0$

## ■ Penalty method

- New functional

$$F_p(\mathbf{x}) = F(\mathbf{x}) + \boxed{\epsilon \langle -g(\mathbf{x}) \rangle^2} = F(\mathbf{x}) + \begin{cases} 0, & \text{if } g(\mathbf{x}) \geq 0 \quad \textit{non-contact} \\ \epsilon g^2(\mathbf{x}), & \text{if } g(\mathbf{x}) < 0 \quad \textit{contact} \end{cases}$$

where  $\epsilon$  is the penalty parameter.

- Stationary point must satisfy

$$\nabla F_p(\mathbf{x}) = \nabla F(\mathbf{x}) + 2\epsilon \langle -g(\mathbf{x}) \rangle \nabla g(\mathbf{x}) = 0$$

- Solution **tends** to the precise solution as  $\epsilon \rightarrow \infty$

## ■ Lagrange multipliers method

## ■ Augmented Lagrangian method

$$\text{Macaulay brackets } \langle x \rangle = \begin{cases} x, & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

# Optimization methods: recall

Functional to be minimized  $F(\mathbf{x})$  under constraint  $g(\mathbf{x}) \geq 0$

■ Penalty method  $F_p(\mathbf{x}) = F(\mathbf{x}) + \epsilon \langle -g(\mathbf{x}) \rangle^2$

■ **Lagrange multipliers method**

- New functional called **Lagrangian**

$$\mathcal{L}(\mathbf{x}, \lambda) = F(\mathbf{x}) + \lambda g(\mathbf{x})$$

- Saddle point problem

$$\min_x \max_{\lambda} \{\mathcal{L}(\mathbf{x}, \lambda)\} \longrightarrow \mathbf{x}^* \longleftarrow \min_{g(\mathbf{x}) \geq 0} \{F(\mathbf{x})\}$$

- Stationary point

$$\nabla_{\mathbf{x}, \lambda} \mathcal{L} = \begin{bmatrix} \nabla_{\mathbf{x}} F(\mathbf{x}) + \lambda \nabla_{\mathbf{x}} g(\mathbf{x}) \\ g(\mathbf{x}) \end{bmatrix} = 0 \quad \text{need to verify } \lambda \leq 0$$

■ **Augmented Lagrangian method**

Macaulay brackets  $\langle x \rangle = \begin{cases} x, & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases}$

# Optimization methods: recall

Functional to be minimized  $F(\mathbf{x})$  under constraint  $g(\mathbf{x}) \geq 0$

■ Penalty method  $F_p(\mathbf{x}) = F(\mathbf{x}) + \epsilon \langle -g(\mathbf{x}) \rangle^2$

■ Lagrange multipliers method  $\mathcal{L}(\mathbf{x}, \lambda) = F(\mathbf{x}) + \lambda g(\mathbf{x})$

■ **Augmented Lagrangian method**

[Hestnes 1969], [Powell 1969], [Glowinski & Le Tallec 1989], [Alart & Curnier 1991], [Simo & Laursen 1992]

• New functional, augmented Lagrangian

$$\mathcal{L}_a(\mathbf{x}, \lambda) = F(\mathbf{x}) + \begin{cases} \lambda g(\mathbf{x}) + \epsilon g^2(\mathbf{x}), & \text{if } \lambda + 2\epsilon g(\mathbf{x}) \geq 0, \text{ contact} \\ -\frac{1}{4\epsilon} \lambda^2, & \text{if } \lambda + 2\epsilon g(\mathbf{x}) < 0, \text{ non-contact} \end{cases}$$

• Stationary point

$$\nabla_{\mathbf{x}, \lambda} \mathcal{L}_a = \begin{cases} \begin{bmatrix} \nabla_{\mathbf{x}} F(\mathbf{x}) + \lambda \nabla_{\mathbf{x}} g(\mathbf{x}) + 2\epsilon g(\mathbf{x}) \nabla g(\mathbf{x}) \\ g(\mathbf{x}) \end{bmatrix} = 0, & \text{if contact} \\ \begin{bmatrix} \nabla_{\mathbf{x}} F(\mathbf{x}) \\ -\frac{\lambda}{\epsilon} \end{bmatrix} = 0, & \text{if non-contact} \end{cases}$$

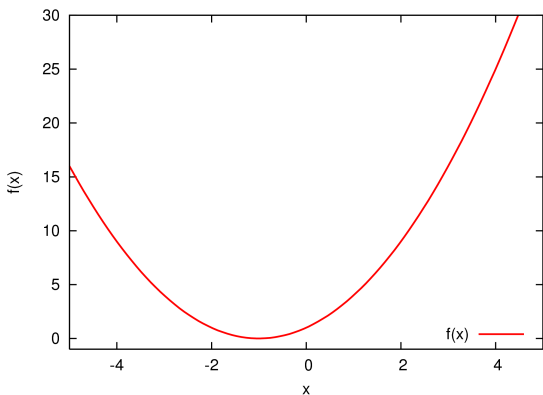
Macaulay brackets  $\langle x \rangle = \begin{cases} x, & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases}$



Uzawa algorithm



# Optimization methods: example

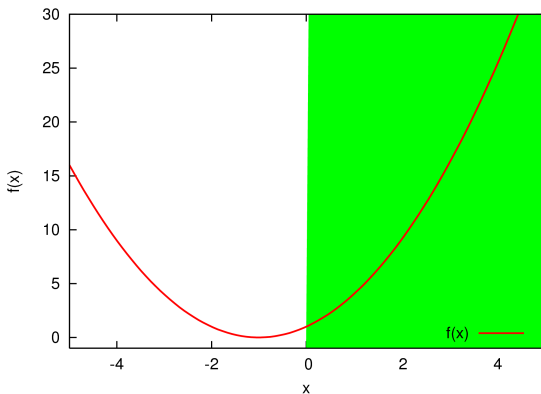


**Functional :**  $f(x) = x^2 + 2x + 1$

**Constrain :**  $g(x) = x \geq 0$

**Solution :**  $x^* = 0$

# Optimization methods: example



Functional :  $f(x) = x^2 + 2x + 1$

**Constrain :**  $g(x) = x \geq 0$

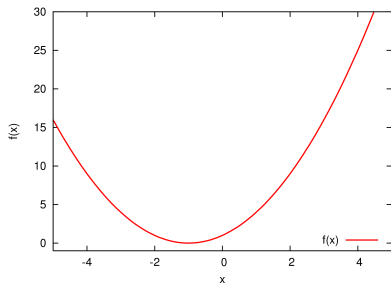
**Solution :**  $x^* = 0$

# Penalty method: example

$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

## ■ Penalty method

$$F_p(x) = F(x) + \epsilon \langle -g(x) \rangle^2$$

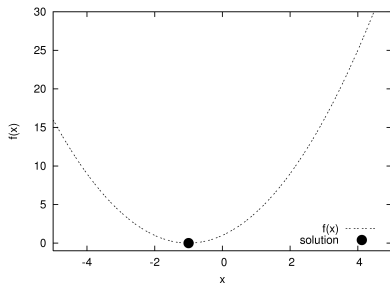
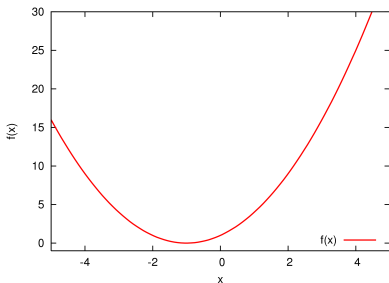


# Penalty method: example

$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

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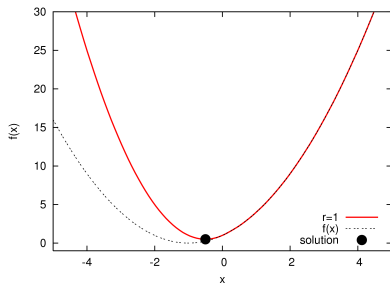
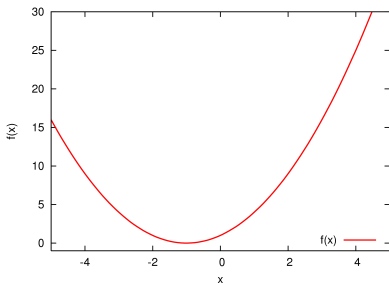
$$\epsilon = 0$$

# Penalty method: example

$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

## ■ Penalty method

$$F_p(x) = F(x) + \epsilon \langle -g(x) \rangle^2$$



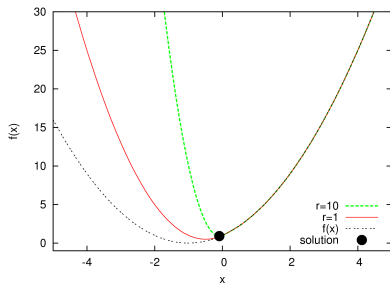
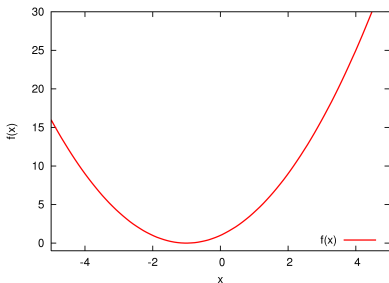
$$\epsilon = 1$$

# Penalty method: example

$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

## ■ Penalty method

$$F_p(x) = F(x) + \epsilon \langle -g(x) \rangle^2$$



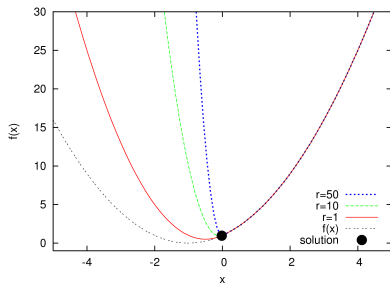
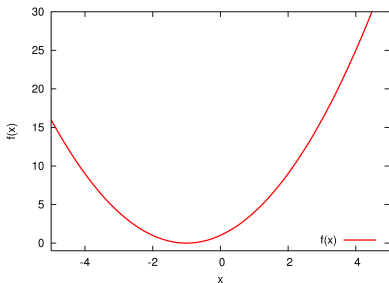
$$\epsilon = 10$$

# Penalty method: example

$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

## ■ Penalty method

$$F_p(x) = F(x) + \epsilon \langle -g(x) \rangle^2$$



$\epsilon = 50$

# Penalty method: example

$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

## ■ Penalty method

$$F_p(x) = F(x) + \epsilon \langle -g(x) \rangle^2$$

### Advantages 😊

- simple physical interpretation
- simple implementation
- no additional degrees of freedom
- “mathematically” smooth functional

### Drawbacks ☹️

- practically non-smooth functional
- solution is not exact:
  - too small penalty  $\rightarrow$  large penetration
  - too large penalty  $\rightarrow$  ill-conditioning of the tangent matrix
- user has to choose penalty  $\epsilon$  properly or automatically and/or adapt during convergence



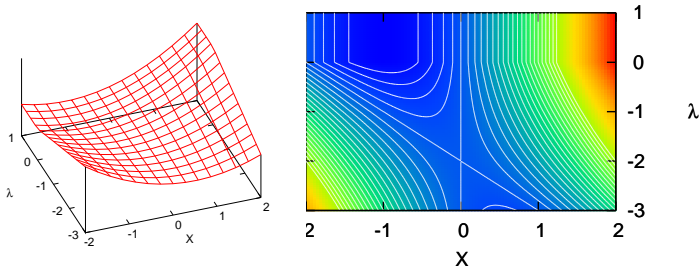
# Lagrange multipliers method: example

$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

## ■ Lagrange multipliers method

$$\mathcal{L}(x, \lambda) = F(x) + \lambda g(x) \rightarrow \text{Saddle point} \rightarrow \min_x \max_{\lambda} \mathcal{L}(x, \lambda)$$

Need to check that  $\lambda \leq 0$



# Lagrange multipliers method: example

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## Advantages ☺

- exact solution
- no adjustable parameters

## Drawbacks ☹

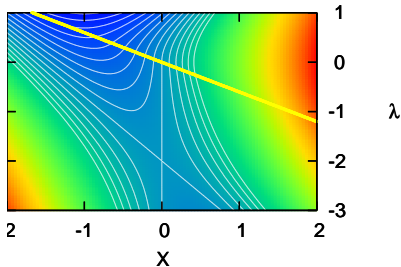
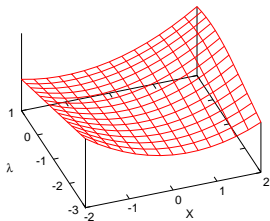
- Lagrangian is not smooth
- additional degrees of freedom
- not fully unconstrained:  $\lambda \leq 0$

# Augmented Lagrangian method: example

$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

## ■ Augmented Lagrangian method

$$\mathcal{L}_a(\mathbf{x}, \lambda) = F(\mathbf{x}) + \begin{cases} \lambda g(\mathbf{x}) + \epsilon g^2(\mathbf{x}), & \text{if } \lambda + 2\epsilon g(\mathbf{x}) \geq 0, \text{ contact} \\ -\frac{1}{4\epsilon} \lambda^2, & \text{if } \lambda + 2\epsilon g(\mathbf{x}) < 0, \text{ non-contact} \end{cases}$$



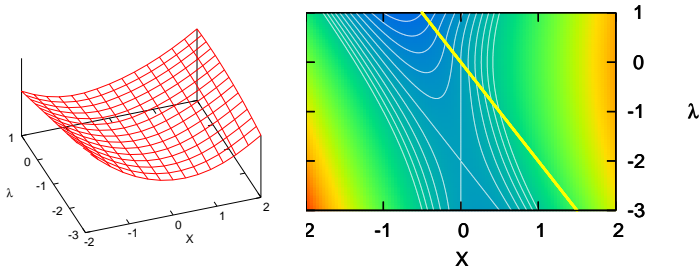
Yellow line separates contact and non-contact regions

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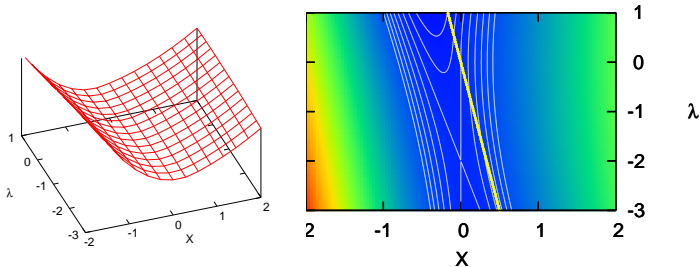
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### Advantages ☺

- exact solution
- smooth functional (!)
- fully unconstrained

### Drawbacks ☹

- additional degrees of freedom
- quite sensitive to parameter  $\epsilon$
- need to adjust  $\epsilon$  during convergence

# Application to contact problems: weak form

$$\int_{\Omega} \underline{\underline{\sigma}} \cdot \delta \nabla \underline{u} d\Omega + \int_{\Gamma_c^1} \underbrace{\boxed{C}}_{\text{Contact term}} d\Gamma_c^1 = \int_{\Gamma_f} \underline{\underline{\sigma}}_0 \cdot \delta \underline{u} d\Gamma + \int_{\Omega} \underline{f}_v \cdot \delta \underline{u} d\Omega,$$

$$\mathbb{K} = \{ \delta \underline{u} \in \mathbb{H}^1(\Omega) \mid \delta \underline{u} = 0 \text{ on } \Gamma_u \}$$

## ■ Penalty method

$$\text{Pressure: } \sigma_n = \epsilon g_n, \text{ Shear: } \underline{\sigma}_t = \begin{cases} \epsilon \underline{g}_t, & \text{if stick } |\sigma_t| < \mu |\sigma_n| \\ \mu \epsilon g_n \delta \underline{g}_t / |\delta \underline{g}_t|, & \text{if slip } |\sigma_t| = \mu |\sigma_n| \end{cases}$$

Contact term

$$C = C(g_n, \underline{g}_t, \delta g_n, \delta \underline{g}_t) = \sigma_n \delta g_n + \underline{\sigma}_t \cdot \delta \underline{g}_t$$

# Application to contact problems: weak form

$$\int_{\Omega} \underline{\underline{\sigma}} \cdot \delta \nabla \underline{u} \, d\Omega + \int_{\Gamma_c^1} \underbrace{\boxed{C}}_{\text{Contact term}} \, d\Gamma_c^1 = \int_{\Gamma_f} \underline{\sigma}_0 \cdot \delta \underline{u} \, d\Gamma + \int_{\Omega} \underline{f}_v \cdot \delta \underline{u} \, d\Omega,$$

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## ■ Augmented Lagrangian method

Contact term

$$C = C(g_n, \underline{g}_t, \lambda_n, \underline{\lambda}_t, \delta g_n, \delta \underline{g}_t, \delta \lambda_n, \delta \underline{\lambda}_t)$$

$$C = \begin{cases} -\frac{1}{\epsilon} (\lambda_n \delta \lambda_n - \underline{\lambda}_t \cdot \delta \underline{\lambda}_t), & \text{if non-contact } \lambda_n + \epsilon g_n \geq 0 \\ \hat{\lambda}_n \delta g_n + g_n \delta \lambda_n + \hat{\underline{\lambda}}_t \cdot \delta \underline{g}_t + \underline{g}_t \cdot \delta \hat{\underline{\lambda}}_t, & \text{if stick } |\hat{\underline{\lambda}}_t| \leq \mu |\hat{\sigma}_n| \\ \hat{\lambda}_n \delta g_n + g_n \delta \lambda_n + \mu \hat{\sigma}_n - \mu \hat{\sigma}_n \frac{\hat{\underline{\lambda}}_t}{|\hat{\underline{\lambda}}_t|} \cdot \delta \underline{g}_t - \frac{1}{\epsilon} \left( \lambda_t + \mu \hat{\sigma}_n \frac{\hat{\underline{\lambda}}_t}{|\hat{\underline{\lambda}}_t|} \right) \cdot \delta \underline{\lambda}_t, & \text{if slip } |\hat{\underline{\lambda}}_t| \geq \mu |\hat{\sigma}_n| \end{cases}$$

where  $\hat{\lambda}_n = \lambda_n + \epsilon g_n$  and  $\hat{\underline{\lambda}}_t = \underline{\lambda}_t + \epsilon \underline{g}_t$ .







The scream

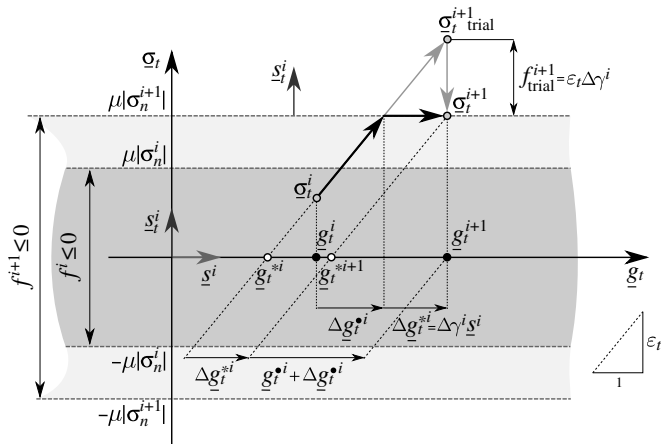
# Friction: methods

- Optimization methods: penalty or augmented Lagrangian method
- Return mapping algorithm for penalty
- Analogy with elasto-plastic formulation problem<sup>[1]</sup>

[1] Curnier "A theory of friction" Int J Solids Struct 20 (1984)

# Friction: Return mapping algorithm

## Return mapping algorithm in 2D

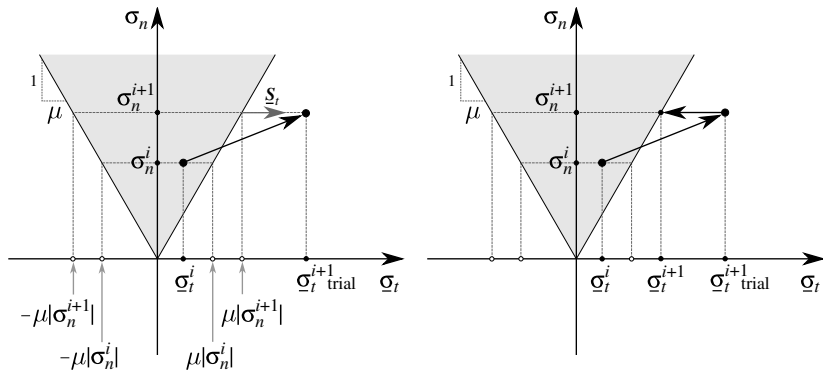


As in plasticity<sup>[1]</sup>

[1] Simo J.C. and Hughes T.J.. Computational inelasticity. Springer (2006)

# Friction: Return mapping algorithm

## ■ Return mapping algorithm in 2D

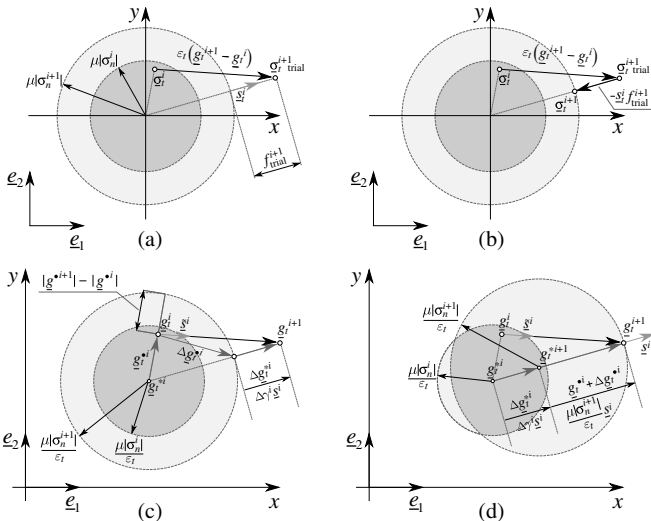


Analogy with non-associated plastic flow<sup>[2]</sup>

[2] Curnier A. A theory of friction. International Journal of Solids and Structures 20 (1984)

# Friction: Return mapping algorithm

## Return mapping algorithm in 3D



A slightly more messy thing

# Application to contact problems: linearization

- Non-linear equation

$$R(\underline{u}, \underline{f}) = 0$$

- Contains  $\delta g_n, \delta g_t$
- Use Newton-Raphson method
- Initial state at step  $i$

$$R(\underline{u}^i, \underline{f}^i) = 0$$

- Should be also satisfied at step  $i + 1$

$$R(\underline{u}^{i+1}, \underline{f}^{i+1}) = R(\underline{u}^i + \delta \underline{u}, \underline{f}^{i+1}) = 0$$

- Linearize

$$R(\underline{u}^i + \delta \underline{u}, \underline{f}^{i+1}) = R(\underline{u}^i, \underline{f}^{i+1}) + \frac{\partial R(\underline{u})}{\partial \underline{u}} \delta \underline{u} = 0$$

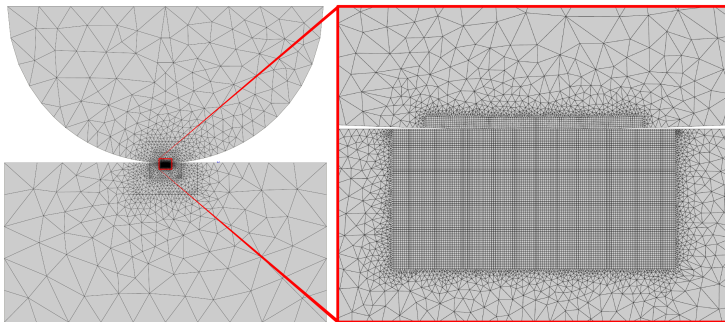
- Finally

$$\delta \underline{u} = - \underbrace{\left[ \frac{\partial R(\underline{u})}{\partial \underline{u}} \right]^{-1}}_{\text{contains } \Delta \delta g_n, \Delta \delta g_t} R(\underline{u}^i)$$

- NB: Contact problem does not satisfy conditions of Kantorovich theorem on the convergence of Newton's method.

# Particularities: mesh and convergence

- Strong **mesh refinement** is required
  - especially at **unknown edges** of contact zones



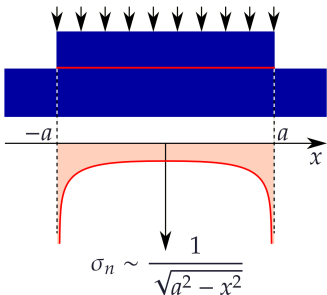
*Typical mesh for fretting analysis [L. Sun, H. Proudhon, G. Cailletaud, 2011]*

*2D ~ 30 000 DoFs, 3D ~ 5 000 000 DoFs*

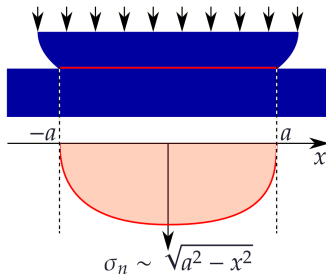


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$$\sigma_n \xrightarrow{x \rightarrow a} -\infty \quad \left| \frac{\partial \sigma_n}{\partial x} \right| \xrightarrow{x \rightarrow a} \infty$$



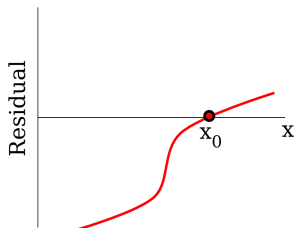
$$\left| \frac{\partial \sigma_n}{\partial x} \right| \xrightarrow{x \rightarrow a} \infty$$

*Infinite contact pressure and/or its derivative*

# Particularities: mesh and convergence

- Strong **mesh refinement** is required
  - especially at **unknown edges** of contact zones
- **Slow change** of boundary conditions:
  - strong non-linearities of contact / friction problems
  - non-uniqueness of solution for frictional problems

## Infinite looping

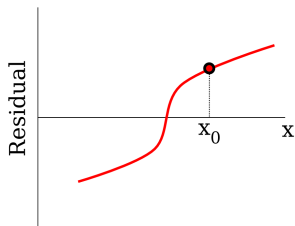


*Initial guess  $R(x_0, f_0) = 0$*

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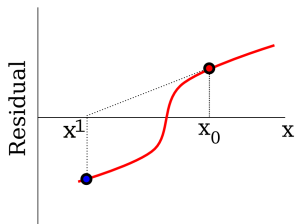


*Too rapid change in boundary conditions  $R(x_0, f_1) \neq 0$*

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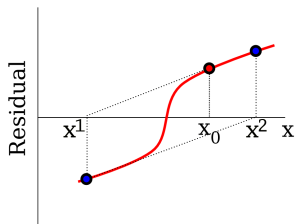
*Iterations of Newton-Raphson method*

$$R(x_0, f_1) + \left. \frac{\partial R}{\partial x} \right|_{x_0} \delta x = 0 \rightarrow \delta x = - \left. \frac{\partial R}{\partial x} \right|_{x_0}^{-1} R(x_0, f_1) \rightarrow x^1 = x_0 + \delta x$$

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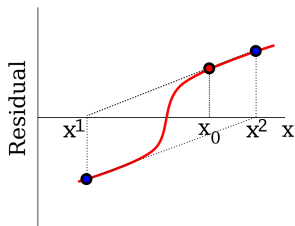
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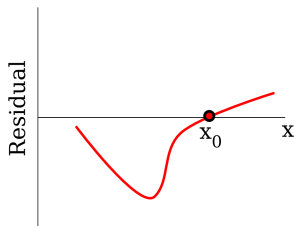


*Infinite looping*

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## Convergence to a “false” solution

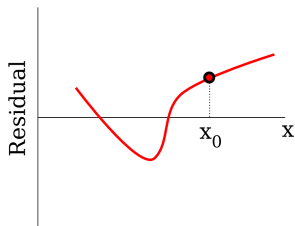


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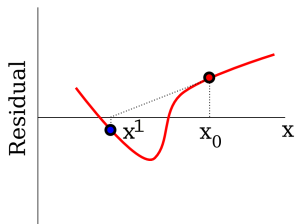
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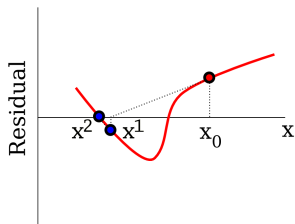
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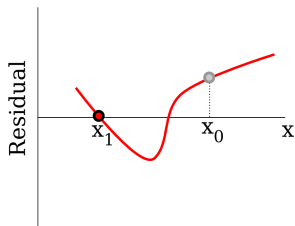
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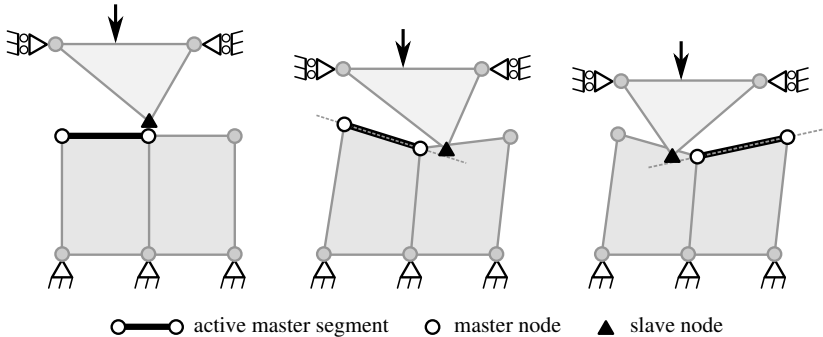
## Convergence to a “false” solution



*Convergence, but is it a “true” solution ?*

# Convergence problems: examples

## ■ Infinite looping, e.g.

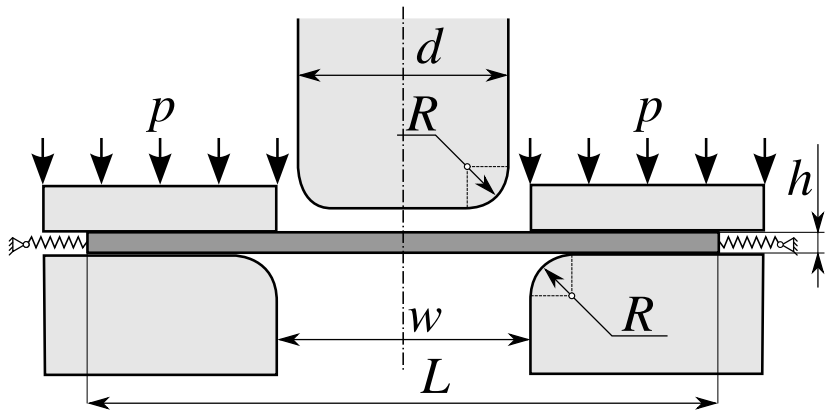


- Change of the contact state (contact/non-contact, stick/slip)
- Interplay between stiffness, friction and augmented Lagrangian coefficients<sup>[1]</sup>
- Combination of non-linearities (e.g., plasticity+contact)

Alart P, Méthode de Newton généralisée en mécanique du contact *Journal de Mathématiques Pures et Appliquées* 76 (1997)

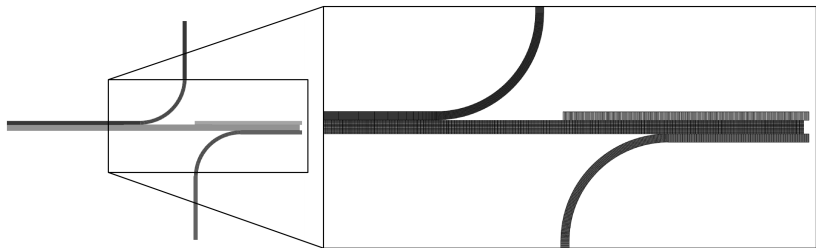
# Convergence problems: examples

- Simulation of a deep drawing problem
- Finite strain plasticity + frictional contact



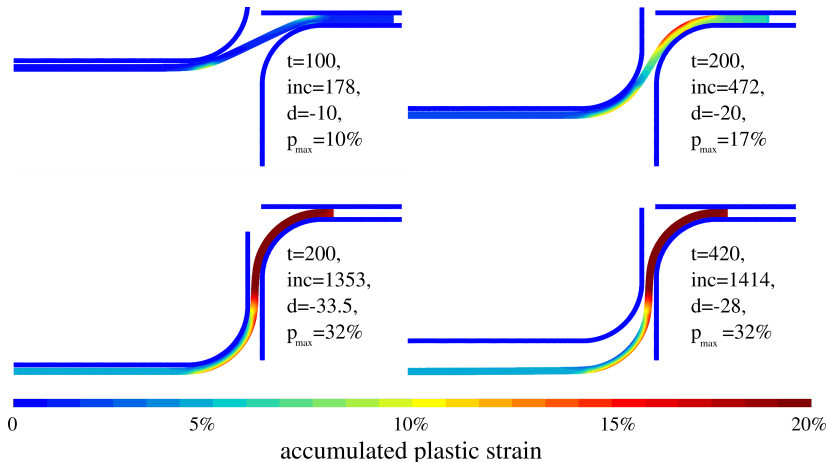
# Convergence problems: examples

- Simulation of a deep drawing problem
- Finite strain plasticity + frictional contact



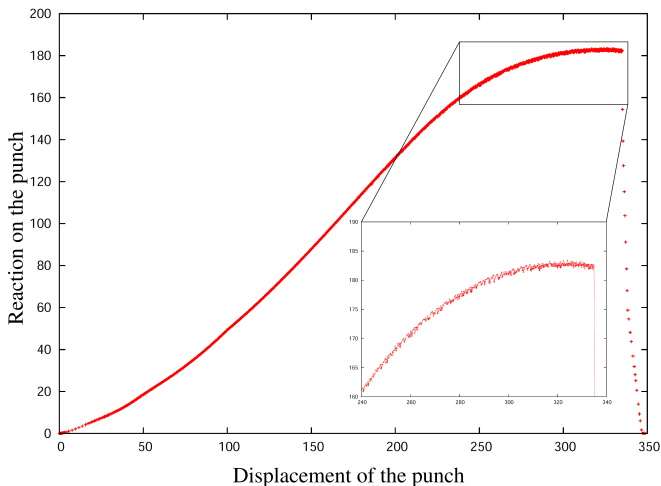
# Convergence problems: examples

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- Dinite strain plasticity + frictional contact



# Convergence problems: examples

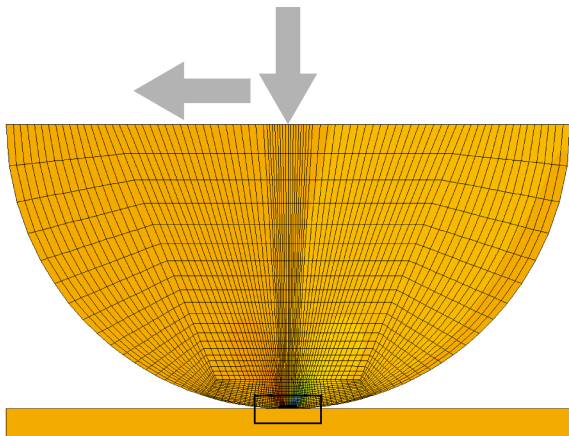
- Simulation of a deep drawing problem
- Dinite strain plasticity + frictional contact





# Cylinder-plane frictional contact

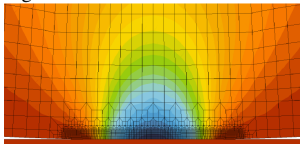
- Non-conservative problem, history of loading is crucial



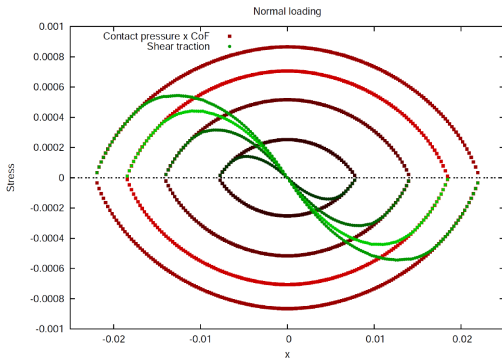
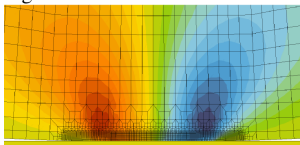
# Cylinder-plane frictional contact

- Non-conservative problem, history of loading is crucial

sig22 at maximal normal load



sig12 at maximal normal load

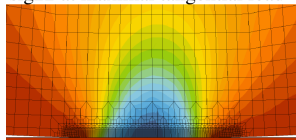


Press in 100 increments,  $u_z \sim t^2$

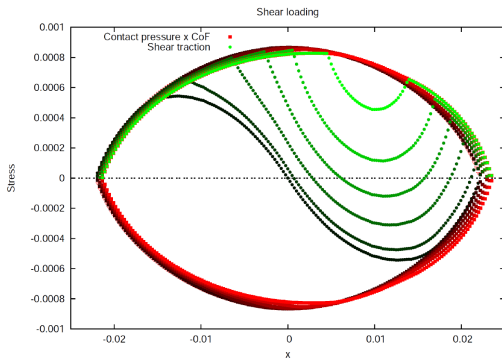
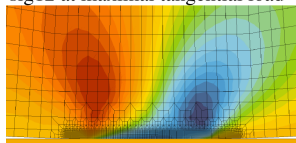
# Cylinder-plane frictional contact

- Non-conservative problem, history of loading is crucial

sig22 at maximal tangential load



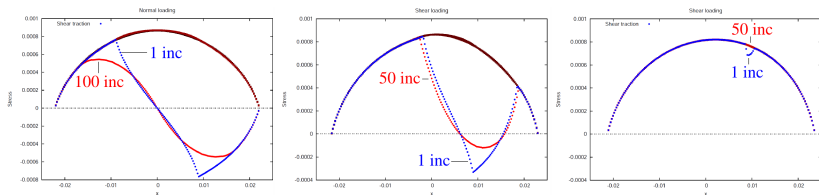
sig12 at maximal tangential load



Shift in 100 increments,  $u_z \sim t$

# Cylinder-plane frictional contact

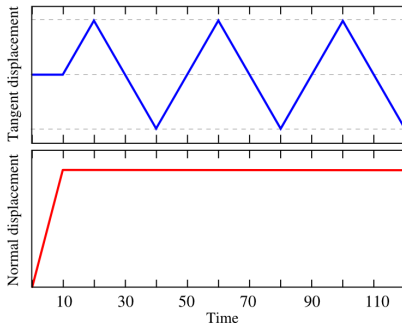
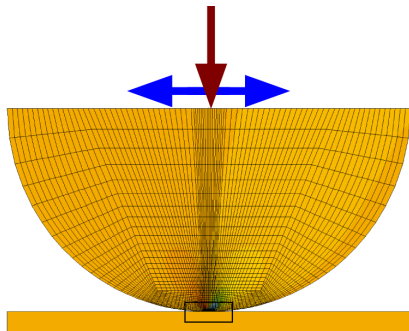
- Non-conservative problem, history of loading is crucial



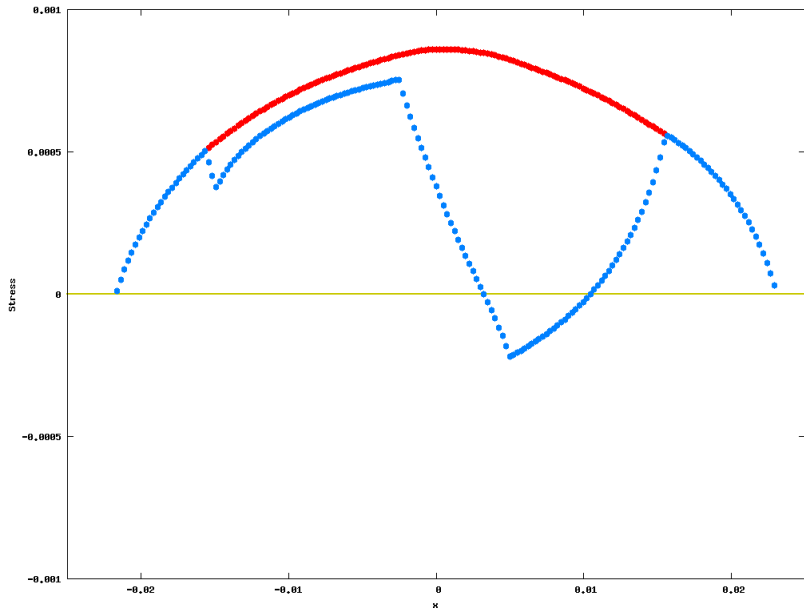
Comparison with: press in 1 increment, shift in 2 increments

*Before stick every point of the contact interface has to pass through the slip zone. It is impossible when loaded too fast.*

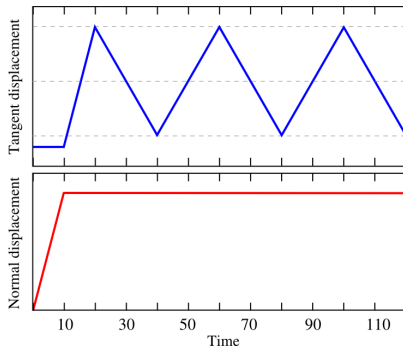
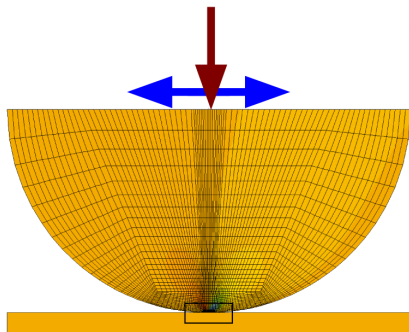
# Sphere-plane frictional contact: cycling



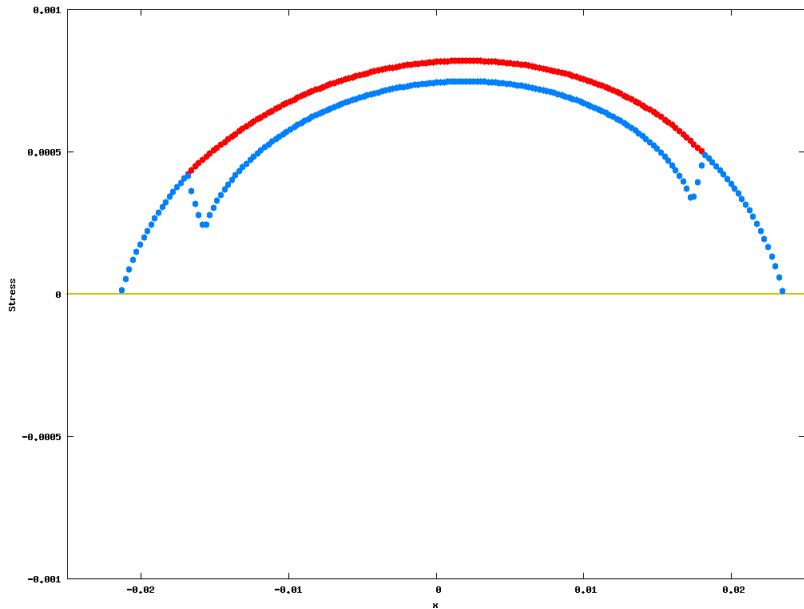
# Sphere-plane frictional contact: cycling



# Sphere-plane frictional contact: cycling



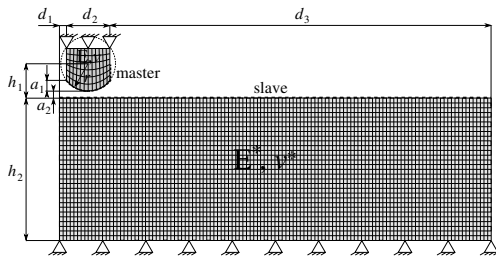
# Sphere-plane frictional contact: cycling





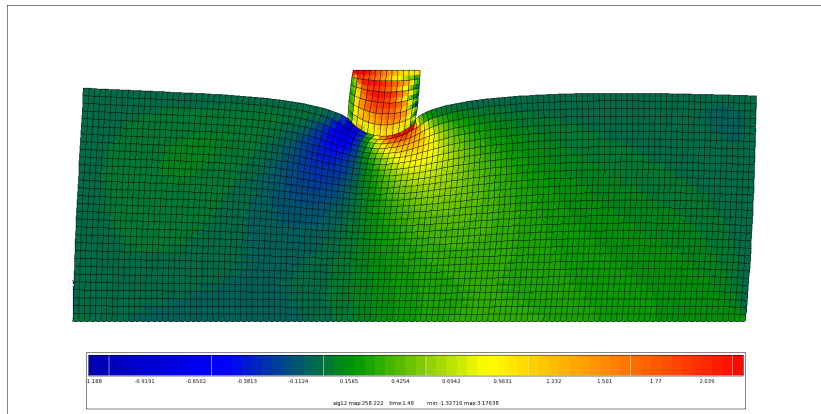
# Shallow ironing test

- Deformable-on-deformable frictional sliding
- Results obtained by different groups<sup>1,2,3,4,5,6</sup> differ significantly
- Local and global friction coefficients may differ



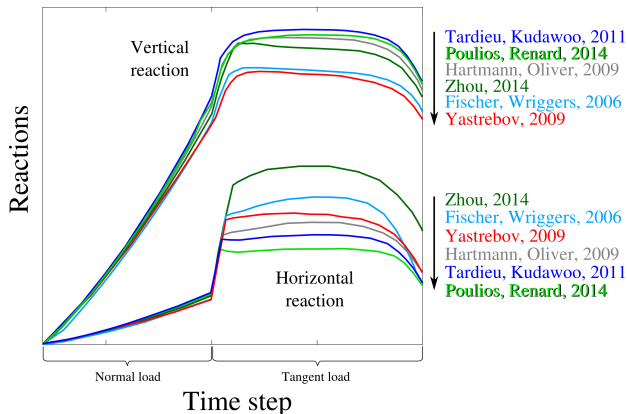
- [1] Fischer K. A., Wriggers P., "Mortar based frictional contact formulation for higher order interpolations using the moving friction cone", *Computer Methods in Applied Mechanics and Engineering*, vol. 195, p. 5020-5036, 2006.
- [2] Hartmann S., Oliver J., Cante J. C., Weyler R., Hernández J. A., "A contact domain method for large deformation frictional contact problems. Part 2: Numerical aspects", *Computer Methods in Applied Mechanics and Engineering*, vol. 198, p. 2607-2631, 2009.
- [3] Yastrebov V. A., "Computational contact mechanics: geometry, detection and numerical techniques", Thèse CdM & Onera, 2011.
- [4] Kudawoo A. D., "Problèmes industriels de grande dimension en mécanique numérique du contact : performance, fiabilité et robustesse", Thèse @ LMA & LAMSID, 2012.
- [5] Poullos K., Renard Y., "A non-symmetric integral approximation of large sliding frictional contact problems of deformable bodies based on ray-tracing", soumis, 2014.
- [6] Zhou Lei's blog, <http://kt2008plus.blogspot.de>

# Shallow ironing test



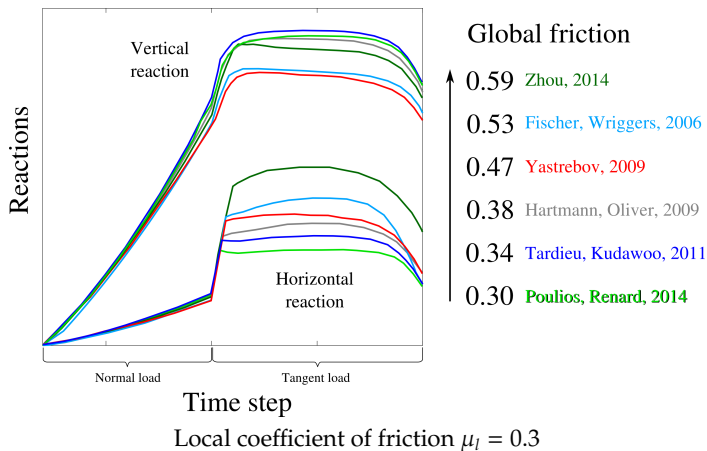
# Shallow ironing test

- No agreement between authors
- Dif. authors used dif. meshes (quadrilateral lin./sq., triangular lin.)
- Dif. authors used either finite or infinitesimal strain formulation



# Shallow ironing test

- No agreement between authors
- Dif. authors used dif. meshes (quadrilateral lin./sq., triangular lin.)
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# Examples of contact problems

## With analytical solution

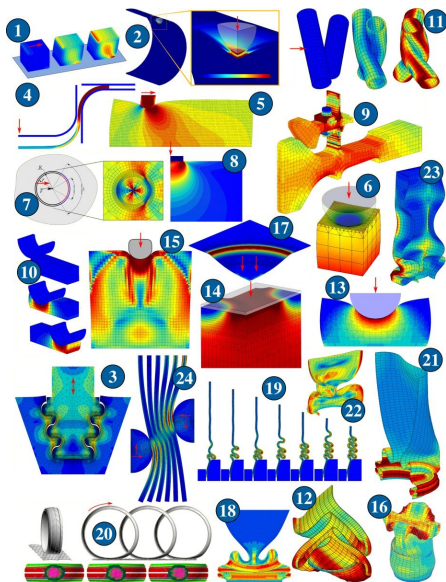
- ★ linear elasticity
- ★ with/without friction

## From literature

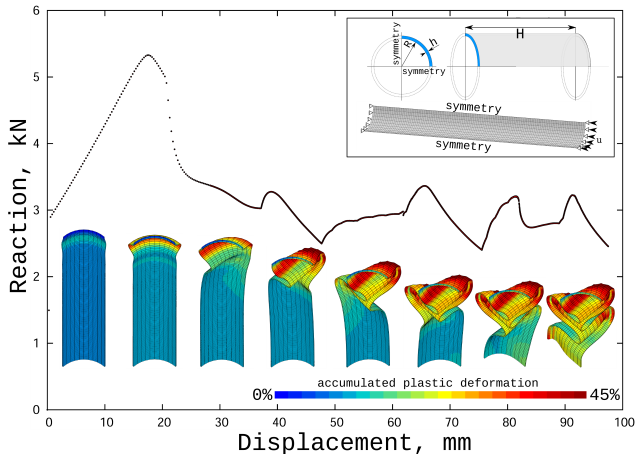
- ★ post-buckling 2D
- ★ finite strains
- ★ elasticity / plasticity
- ★ with/without friction

## New

- ★ multi-contacts
- ★ post-buckling 3D
- ★ finite strains
- ★ elasticity / plasticity
- ★ with/without friction



# Self-contact problem



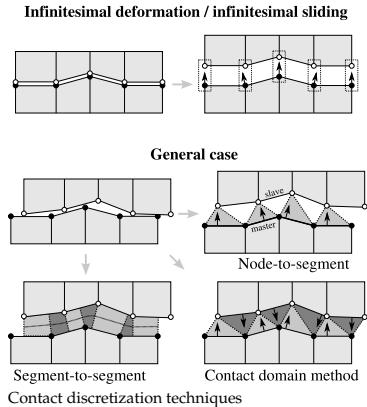
*Finite element analysis of a post-buckling behavior of a thin walled tube*

*Collection of non-linearities: buckling instability, self-contact, finite strain plasticity*



- It's just a tip of the "Computational Contact Mechanics" iceberg
- Contact detection
- Contact discretization and integration
- Smoothing techniques
- Energy conservative methods for dynamics

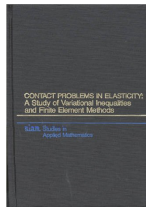
See Basava R. Akula presentation on Friday



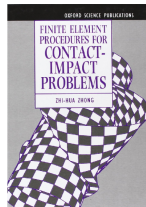
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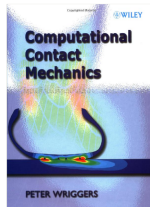
Kikuchi, Oden (1988)



Zhong (1993)



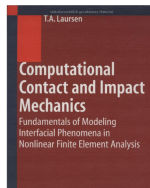
Wriggers (2002)



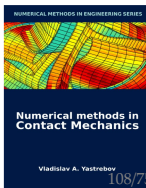
Wriggers, 2<sup>nd</sup> ed. (2006)



Yastrebov (2002)



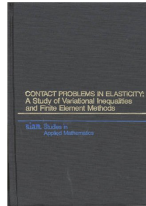
Yastrebov (2013)



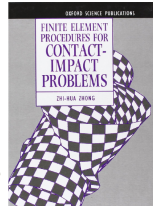


- It's just a tip of the “Computational Contact Mechanics” iceberg
  - Contact detection
  - Contact discretization and integration
  - Smoothing techniques
  - Energy conservative methods for dynamics
  - Several advanced topics
- see Yastrebov\_CEMEF.pdf, page 18.

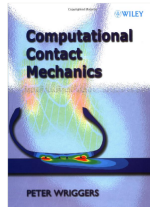
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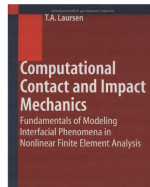
Wriggers (2002)



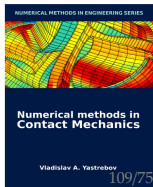
Wriggers, 2<sup>nd</sup> ed. (2006)



Yastrebov (2002)



Yastrebov (2013)



$\mathcal{L}_a(x, \lambda)$

Thank you for your attention!

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