Computational Approach to Micromechanical Contacts

Lecture 1.
Introduction to the Finite Element Method

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Outline

Elements of tensor algebra

Vectors and tensors

Tensor notations

- Scalars $\in \mathbb{R}$:
  a, α, C

Component notations

- Scalars $\in \mathbb{R}$:
  a, α, C

Vectors and tensors

Tensor notations

- Scalars $\in \mathbb{R}$:
  a, α, C
- Vectors $\in \mathbb{V}_{\text{dim}}$:
  $\underline{\boldsymbol{a}}, \underline{\boldsymbol{\tau}}$

Component notations

- Scalars $\in \mathbb{R}$:
  a, α, C
- Vectors* $\in \mathbb{R}^{\text{dim}}$:
  a_i, τ_j
  with $\underline{\boldsymbol{a}} = a_i \underline{\boldsymbol{e}}^i$ and $a_i = \underline{\boldsymbol{e}}^i \cdot \underline{\boldsymbol{a}}$

**Component notations require introducing a basis $\underline{\boldsymbol{e}}^i, i = 1 \dots \text{dim}$ and a dual basis $\underline{\boldsymbol{e}}_j$ such that $\underline{\boldsymbol{e}}_j \cdot \underline{\boldsymbol{e}}^i = \delta^i_j$, where $\delta^i_j = 0$ if $i \neq j$ and $\delta^i_j = 1$ if $i = j$.*

Vectors and tensors

Tensor notations

- Scalars $\in \mathbb{R}$:
  a, α, C
- Vectors $\in \mathbb{V}_{\text{dim}}$:
  $\underline{\boldsymbol{a}}, \underline{\boldsymbol{\tau}}$
- Second-order tensors $\in \mathbb{T}^2_{\text{dim}}$:
  $\underline{\underline{\boldsymbol{A}}}, \underline{\underline{\boldsymbol{\sigma}}}$

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  a_i, τ_j
  with $\underline{\boldsymbol{a}} = a_i \underline{\boldsymbol{e}}^i$ and $a_i = \underline{\boldsymbol{e}}^i \cdot \underline{\boldsymbol{a}}$
- Second-order tensors $\in \mathbb{R}^{\text{dim}} \times \mathbb{R}^{\text{dim}}$:
  A_{ij}, σ_{kl}
  with $\underline{\underline{\boldsymbol{A}}} = A_{ij} \underline{\boldsymbol{e}}^i \otimes \underline{\boldsymbol{e}}^j$ and $A_{ij} = \underline{\boldsymbol{e}}_i \cdot \underline{\underline{\boldsymbol{A}}} \cdot \underline{\boldsymbol{e}}_j$

**Component notations require introducing a basis* $\underline{\boldsymbol{e}}^i, i = 1 \dots \text{dim}$ *and a dual basis* $\underline{\boldsymbol{e}}_j$ *such that* $\underline{\boldsymbol{e}}_j \cdot \underline{\boldsymbol{e}}^i = \delta^i_j$*, where* $\delta^i_j = 0$ *if* $i \neq j$ *and* $\delta^i_j = 1$ *if* $i = j$.

Vectors and tensors

Tensor notations

- Scalars $\in \mathbb{R}$:
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- Forth-order tensors $\in \mathbb{T}^4_{\mathrm{dim}}$:
  ${}^4\underline{\underline{\boldsymbol{C}}}$

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- Forth-order tensors
  $\in \mathbb{R}^{\mathrm{dim}} \times \dots \times \mathbb{R}^{\mathrm{dim}}$:
  C_{ijkl}
  with ${}^4\underline{\underline{\boldsymbol{C}}} = C_{ijkl} \underline{\boldsymbol{e}}^i \otimes \underline{\boldsymbol{e}}^j \otimes \underline{\boldsymbol{e}}^k \otimes \underline{\boldsymbol{e}}^l$ and
  $C_{ijkl} = \underline{\boldsymbol{e}}_l \cdot \left(\underline{\boldsymbol{e}}_k \cdot \left(\underline{\boldsymbol{e}}_j \cdot \left(\underline{\boldsymbol{e}}_i \cdot {}^4\underline{\underline{\boldsymbol{C}}}\right)\right)\right)$

**Component notations require introducing a basis $\underline{\boldsymbol{e}}^i, i = 1 \dots \mathrm{dim}$ and a dual basis $\underline{\boldsymbol{e}}_j$ such that $\underline{\boldsymbol{e}}_j \cdot \underline{\boldsymbol{e}}^i = \delta^i_j$, where $\delta^i_j = 0$ if $i \neq j$ and $\delta^i_j = 1$ if $i = j$.*

Tensor algebra

Tensor notations

- Transposition
  $$\underline{\underline{C}} = \underline{\underline{D}}^\intercal,\ \left(\underline{\underline{A}}\cdot\underline{\underline{B}}\right)^\intercal = \underline{\underline{B}}^\intercal\cdot\underline{\underline{A}}^\intercal$$
- Symmetric tensor
  $$\underline{\underline{A}}^\intercal = \underline{\underline{A}}$$
- Antisymmetric* tensor
  $$\underline{\underline{B}}^\intercal = -\underline{\underline{B}}$$
- Tensor decomposition
  $$\underline{\underline{C}} = \underline{\underline{C}}^S + \underline{\underline{C}}^A \quad \text{with}$$
  $$\underline{\underline{C}}^S = \frac{1}{2}\left(\underline{\underline{C}} + \underline{\underline{C}}^\intercal\right),\ \underline{\underline{C}}^A = \frac{1}{2}\left(\underline{\underline{C}} - \underline{\underline{C}}^\intercal\right)$$

Component notations

- Transposition
  $$C_{ij} = D_{ji}$$
- Symmetric tensor
  $$A_{ij} = A_{ji}$$
- Antisymmetric tensor
  $$B_{ij} = -B_{ji}$$
- Tensor decomposition
  $$C_{ij} = C^S_{ij} + C^A_{ij} \quad \text{with}$$
  $$C^S_{ij} = \frac{1}{2}\left(C_{ij} + C_{ji}\right),\ C^A_{ij} = \frac{1}{2}\left(C_{ij} - C_{ji}\right)$$

Examples

- Identity tensor (symmetric) $\underline{\underline{I}} = \delta^{ij}\underline{e}_i\otimes\underline{e}_j = \underline{e}_i\otimes\underline{e}_i$
- Rotation tensor (asymmetric = symmetric($\neq 0$) + antisymmetric($\neq 0$)):

$$\underline{\underline{Q}} \sim \begin{bmatrix}\cos(\phi) & \sin(\phi) & 0\\ -\sin(\phi) & \cos(\phi) & 0\\ 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}\cos(\phi) & 0 & 0\\ 0 & \cos(\phi) & 0\\ 0 & 0 & 1\end{bmatrix} + \begin{bmatrix}0 & \sin(\phi) & 0\\ -\sin(\phi) & 0 & 0\\ 0 & 0 & 1\end{bmatrix}$$

* *Antisymmetric* $\equiv$ *skew-symmetric*

Tensor algebra: products

Tensor notations

- Mutliplication by a scalar
  $\alpha \underline{\underline{A}} = \underline{\underline{A}} \alpha$
- Scalar product*
  $\underline{a} \cdot \underline{b} = c$
  $\underline{a} \cdot \underline{\underline{A}} = \underline{b}$
  $\underline{\underline{A}} \cdot \underline{\underline{B}} = \underline{\underline{C}}$
- Tensor contraction
  $\underline{\underline{A}} : \underline{\underline{B}} = c$
  $\underline{\underline{A}} \cdot\cdot \underline{\underline{B}} = d$

Component notations

- Mutliplication by a scalar
  $\alpha A_{ij} = A_{ij} \alpha$
- Scalar (dot) product**
  $a_i b^i = c$
  $a_i A^{ij} = b^j$
  $A_{ij} B^{jk} = C_i^k$
- Tensor contraction
  $A_{ij} B_{ij} = c$
  $A_{ij} B_{ji} = d$

- Remark:
  $\underline{\underline{A}} : \underline{\underline{B}} = \underline{\underline{A}}^S : \underline{\underline{B}}^S + \underline{\underline{A}}^A : \underline{\underline{B}}^A$ and $\underline{\underline{A}}^S : \underline{\underline{B}}^A = \underline{\underline{A}}^A : \underline{\underline{B}}^S = 0$
  $\underline{\underline{A}} \cdot\cdot \underline{\underline{B}} = \underline{\underline{A}}^S \cdot\cdot \underline{\underline{B}}^S + \underline{\underline{A}}^A \cdot\cdot \underline{\underline{B}}^A$ and $\underline{\underline{A}}^S \cdot\cdot \underline{\underline{B}}^A = \underline{\underline{A}}^A \cdot\cdot \underline{\underline{B}}^S = 0$

**Scalar product ≡ dot product ≡ inner product.*

***We assume Einstein summation by repeating index, i.e.* $a_i A^{ij} = \sum_{i=1}^{\dim} a_i A^{ij}$.

Tensor algebra: products II & invariants

Tensor notations

- Vector product*

  $\underline{a} \times \underline{b} = \underline{c}$

  such that $\underline{c} \cdot \underline{a} = 0$, $\underline{c} \cdot \underline{b} = 0$

- Tensor product**

  $\underline{a} \otimes \underline{b} = \underline{\underline{C}}$

  $\underline{\underline{A}} \otimes \underline{\underline{B}} = {}^4\underline{\underline{C}}$

- Invariants:

  $I_1(\underline{\underline{A}}) = \mathrm{tr}(\underline{\underline{A}}) = \underline{\underline{I}} : \underline{\underline{A}}$

  $I_2(\underline{\underline{A}}) = \frac{1}{2}\left[\mathrm{tr}(\underline{\underline{A}})^2 - \mathrm{tr}(\underline{\underline{A}}^2)\right]$

  $I_3(\underline{\underline{A}}) = \det(\underline{\underline{A}})$

Component notations

- Vector product*

  $c^i = \epsilon_{ijk} a^j b^k$

  with ϵ_{ijk} Levi-Civita symbol

$$\epsilon_{ijk} = \begin{cases} 1, & \text{if } (i,j,k) = (1,2,3) \text{ or } (2,3,1) \text{ or } (3,1,2) \\ -1, & \text{if } (i,j,k) = (2,1,3) \text{ or } (1,3,2) \text{ or } (3,2,1) \\ 0, & \text{otherwise.} \end{cases}$$

- Tensor product

  $a_i b_j = C_{ij}$

  $A_{ij} B_{kl} = C_{ijkl}$

- Invariants:

  $I_1(\underline{\underline{A}}) = A_{ii} = A_{11} + A_{22} + A_{33}$

  $I_2(\underline{\underline{A}}) = \dots$

  $I_3(\underline{\underline{A}}) = \dots$

Defined only for $\dim = 3$, also called cross product.

**Also called outer product.*

Tensor algebra: deviatoric & spherical parts

Tensor notations

- Spherical part of tensor $\underline{\underline{A}}$

$$\mathrm{Sp}(\underline{\underline{A}}) = \frac{1}{3}\mathrm{tr}(\underline{\underline{A}})\underline{\underline{I}}$$

- Deviatoric part of tensor $\underline{\underline{A}}$

$$\mathrm{Dv}(\underline{\underline{A}}) = \underline{\underline{A}} - \frac{1}{3}\mathrm{tr}(\underline{\underline{A}})\underline{\underline{I}}$$

Component notations

- Spherical part of tensor $\underline{\underline{A}}$

$$\mathrm{Sp}(\underline{\underline{A}}) = \frac{1}{3}(A_{kk})\delta_{ij}$$

- Deviatoric part of tensor $\underline{\underline{A}}$

$$\mathrm{Dv}(\underline{\underline{A}}) = A_{ij} - \frac{1}{3}(A_{kk})\delta_{ij}$$

- Tensor decomposition

$$\underline{\underline{A}} = \mathrm{Sp}(\underline{\underline{A}}) + \mathrm{Dv}(\underline{\underline{A}})$$

- Remark: for an antisymmetric tensor $\underline{\underline{B}}^A$

$$\mathrm{Sp}(\underline{\underline{B}}^A) = 0 \quad \Rightarrow \quad \underline{\underline{B}}^A = \mathrm{Dv}(\underline{\underline{B}}^A)$$

Tensor algebra: principal values

- Principal values of a linear operator $\underline{\underline{A}}$:

$$\underline{\underline{A}} \cdot \underline{u} = \lambda \underline{u} \quad \Leftrightarrow \quad \left(\underline{\underline{A}} - \lambda \underline{\underline{I}}\right) \cdot \underline{u} = 0$$

  If $\underline{\underline{A}} = \underline{\underline{A}}^S$ for $\dim = 3$ then exist three real λ_i and corresponding $\underline{u}_i$ called eigen values and eigen vectors of operator $\underline{\underline{A}}$, respectively. Moreover, for $i \neq j$, $\underline{u}_i \cdot \underline{u}_j = 0$.

- To find λ_i we solve

$$I_3(\underline{\underline{A}}) - I_2(\underline{\underline{A}})\lambda + I_1(\underline{\underline{A}})\lambda^2 - \lambda^3 = 0$$

- Then tensor can be rewritten in its eigen basis:

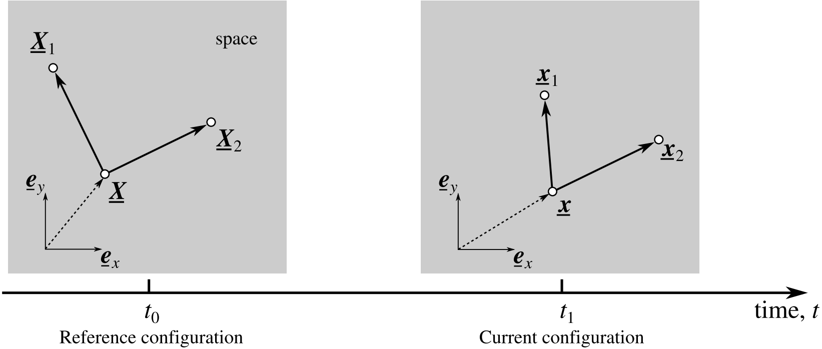
$$\underline{\underline{A}} = \lambda_1 \underline{u}_1 \otimes \underline{u}_1 + \lambda_2 \underline{u}_2 \otimes \underline{u}_2 + \lambda_3 \underline{u}_3 \otimes \underline{u}_3$$

  and $\mathrm{tr}(\underline{\underline{A}}) = \lambda_i |\underline{u}_i|^2$.

Continuum Mechanics: Recall

Deformable medium

- Consider change in positions of points with time t
- Consider two states: $t = t_0$ (reference) and $t = t_1$ (current configurations)
- Point $\underline{X}$ from the reference configuration is labeled $\underline{x}$ in the current configuration
- Displacement vector between t_0 and t_1 is $\underline{u} = \underline{x} - \underline{X}$



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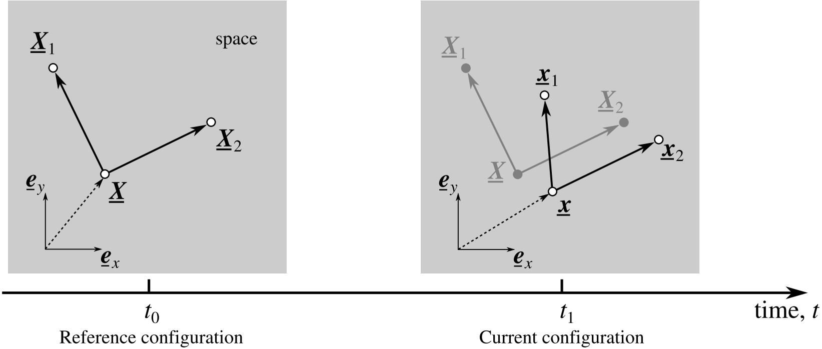

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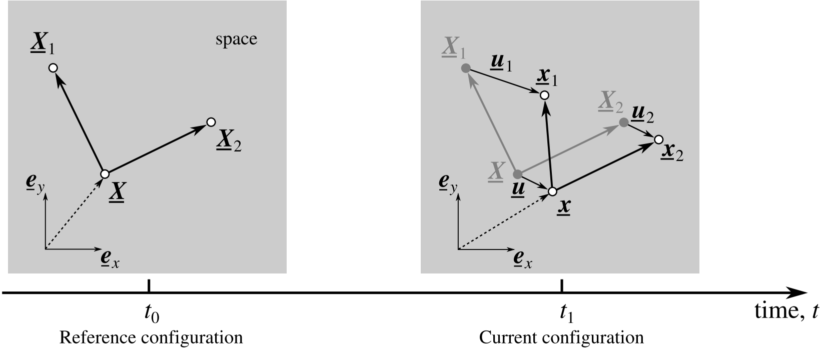
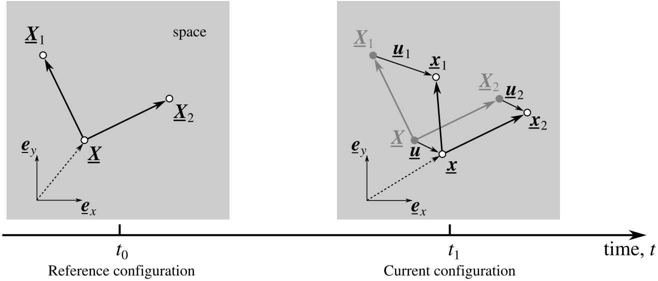


- Transformation gradient $\underline{\underline{F}} = \dfrac{\partial \underline{\boldsymbol{x}}}{\partial \underline{\boldsymbol{X}}} = \dfrac{\partial\left(\underline{\boldsymbol{X}} + \underline{\boldsymbol{u}}\right)}{\partial \underline{\boldsymbol{X}}} = \underline{\underline{I}} + \dfrac{\partial \underline{\boldsymbol{u}}}{\partial \underline{\boldsymbol{X}}} = \underline{\underline{I}} + \underline{\underline{H}}$
- Cauchy-Green right tensor $\underline{\underline{C}} = \underline{\underline{F}}^{\intercal} \cdot \underline{\underline{F}}$
- Green-Lagrange deformation tensor $\underline{\underline{E}} = \dfrac{1}{2}\left(\underline{\underline{C}} - \underline{\underline{I}}\right) = \underline{\underline{H}}^S + \dfrac{1}{2}\underline{\underline{H}}^{\intercal} \cdot \underline{\underline{H}}$
- For $H_{ij} \ll 1$, $\underline{\underline{E}} \approx \underline{\underline{H}}^S$ and we obtain a tensor of small deformations

$$\underline{\underline{\varepsilon}} = \underline{\underline{H}}^S = \frac{1}{2}\left[\frac{\partial \underline{\boldsymbol{u}}}{\partial \underline{\boldsymbol{X}}} + \left(\frac{\partial \underline{\boldsymbol{u}}}{\partial \underline{\boldsymbol{X}}}\right)^{\intercal}\right] = \frac{1}{2}\left(\nabla \underline{\boldsymbol{u}} + (\nabla \underline{\boldsymbol{u}})^{\intercal}\right)$$



Stress tensor and Hooke's law

- Hooke's law in uniaxial test:

$$\boxed{\sigma_{xx} = E\varepsilon_{xx}}$$

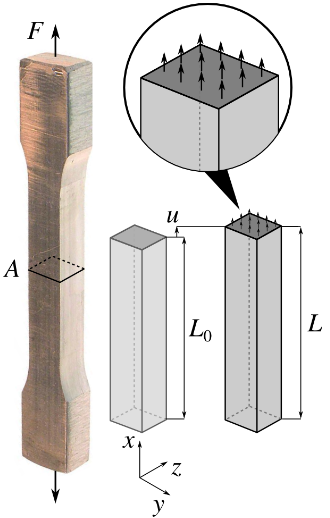
$$F = ku \quad \Leftrightarrow \quad \sigma_{xx}A = \frac{EA}{L_0}u = EA\frac{L-L_0}{L_0}$$

- In general case stress and strain are related through a linear operator (fourth-order elasticity tensor ${}^4\underline{\underline{C}}$):

$$\boxed{\underline{\underline{\sigma}} = {}^4\underline{\underline{C}} : \underline{\underline{\varepsilon}}}$$

- Inversely the strain can be found through a stiffness tensor ${}^4\underline{\underline{S}}$:

$$\boxed{\underline{\underline{\varepsilon}} = {}^4\underline{\underline{S}} : \underline{\underline{\sigma}}}$$



Hooke's law for isotropic solids: stress

- In the case of isotropic material the Hooke's law reduces to:

$$\boxed{\underline{\underline{\sigma}} = \lambda \mathrm{tr}(\underline{\underline{\varepsilon}})\underline{\underline{I}} + 2\mu\underline{\underline{\varepsilon}},}$$

 with λ, μ being Lamé coefficients:

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)}$$

 with Young's modulus E and Poisson's ratio ν.

- In the component form it reads:

$$\sigma_{ij} = \lambda(\varepsilon_{kk})\delta_{ij} + 2\mu\varepsilon_{ij}$$

- In the matrix form:

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} = 2\mu \begin{bmatrix} \lambda \mathrm{tr}(\underline{\underline{\varepsilon}})/(2\mu) + \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \lambda \mathrm{tr}(\underline{\underline{\varepsilon}})/(2\mu) + \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \lambda \mathrm{tr}(\underline{\underline{\varepsilon}})/(2\mu) + \varepsilon_{33} \end{bmatrix}$$

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Hooke's law for isotropic solids: strain

- Strain as a function of stress:

$$\boxed{\underline{\underline{\varepsilon}} = \frac{1+\nu}{E}\underline{\underline{\sigma}} - \frac{\nu}{E}\mathrm{tr}(\underline{\underline{\sigma}})\underline{\underline{\boldsymbol{I}}}}.$$

- In the component form it reads:

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}$$

- In the matrix form:

$$\begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} \end{bmatrix} = \frac{1}{E}\begin{bmatrix} (1+\nu)\sigma_{11} - \nu\mathrm{tr}(\underline{\underline{\sigma}}) & (1+\nu)\sigma_{12} & (1+\nu)\sigma_{13} \\ (1+\nu)\sigma_{12} & (1+\nu)\sigma_{22} - \nu\mathrm{tr}(\underline{\underline{\sigma}}) & (1+\nu)\sigma_{23} \\ (1+\nu)\sigma_{13} & (1+\nu)\sigma_{23} & (1+\nu)\sigma_{33} - \nu\mathrm{tr}(\underline{\underline{\sigma}}) \end{bmatrix}$$

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- In the component form it reads:

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}$$

- In the matrix form:

$$\begin{bmatrix}\varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13}\\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23}\\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33}\end{bmatrix} = \frac{1}{E}\begin{bmatrix}(1+\nu)\sigma_{11} - \nu\mathrm{tr}(\underline{\underline{\sigma}}) & (1+\nu)\sigma_{12} & (1+\nu)\sigma_{13}\\ (1+\nu)\sigma_{12} & (1+\nu)\sigma_{22} - \nu\mathrm{tr}(\underline{\underline{\sigma}}) & (1+\nu)\sigma_{23}\\ (1+\nu)\sigma_{13} & (1+\nu)\sigma_{23} & (1+\nu)\sigma_{33} - \nu\mathrm{tr}(\underline{\underline{\sigma}})\end{bmatrix}$$

$$= \frac{1}{E}\begin{bmatrix}\sigma_{11} - \nu(\sigma_{22}+\sigma_{33}) & (1+\nu)\sigma_{12} & (1+\nu)\sigma_{13}\\ (1+\nu)\sigma_{12} & \sigma_{22} - \nu(\sigma_{11}+\sigma_{33}) & (1+\nu)\sigma_{23}\\ (1+\nu)\sigma_{13} & (1+\nu)\sigma_{23} & \sigma_{33} - \nu(\sigma_{11}+\sigma_{22})\end{bmatrix}$$

Equilibrium of an infinitesimal element

- Infinitesimal strain tensor is symmetric and satisfies the compatibility conditions*:

  $$\nabla \times \left(\nabla \times \underline{\underline{\varepsilon}}\right) = 0$$

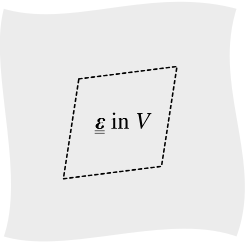
- Stress tensor $\underline{\underline{\sigma}}$ should ensure equilibrium of infinitesimal element**:

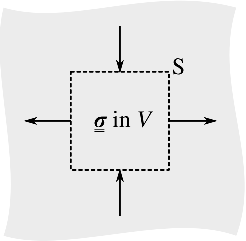
  Force balance: $\int\limits_S \underline{n} \cdot \underline{\underline{\sigma}} \, dS = 0$

  Momentum balance: $\int\limits_S \underline{r} \times (\underline{n} \cdot \underline{\underline{\sigma}}) \, dS = 0$

- Following Gauss-Ostrogradsky theorem:

  $\int\limits_S \underline{n} \cdot \underline{\underline{\sigma}} \, dS = \int\limits_V \nabla \cdot \underline{\underline{\sigma}} \, dV = 0$ Since volume V can be arbitrary chosen, then $\boxed{\nabla \cdot \underline{\underline{\sigma}} = 0}$ everywhere in V.





*In case of a simply-connected solid.
**In absence of volumetric forces.

Equilibrium of an infinitesimal element II

- Second Newton's law:

$$m\underline{\ddot{u}} = \underline{f} \quad \Rightarrow \quad \rho\underline{\ddot{u}} = \frac{1}{V}\underline{f}$$

- In presence of volumetric forces with density $\underline{f}_V$, the total force is given by:

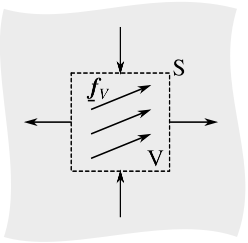
$$\underline{f} = \int\limits_V \underline{f}_V \, dV + \int\limits_S \underline{n} \cdot \underline{\underline{\sigma}} \, dS$$

- Then using the second Newton's law and Gauss-Ostrogradsky's theorem:

$$\int\limits_V \left(\nabla \cdot \underline{\underline{\sigma}} + \underline{f}_V \right) dV = \int\limits_V \rho\underline{\ddot{u}} \, dV$$

- Since it is right for arbitrary V, then in every point of V:

$$\boxed{\nabla \cdot \underline{\underline{\sigma}} + \underline{f}_V = \rho\underline{\ddot{u}}}$$



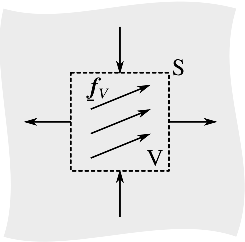
Equilibrium of an infinitesimal element II

- Equilibrium (3 equations):

$$\nabla \cdot \underline{\underline{\sigma}} + \underline{f}_V = \rho \ddot{\underline{u}}$$

- In component form:

$$\frac{\partial \sigma_{11}}{\partial x} + \frac{\partial \sigma_{12}}{\partial y} + \frac{\partial \sigma_{13}}{\partial z} + f_{V_x} = \rho \ddot{u}_x$$
$$\frac{\partial \sigma_{12}}{\partial x} + \frac{\partial \sigma_{22}}{\partial y} + \frac{\partial \sigma_{23}}{\partial z} + f_{V_y} = \rho \ddot{u}_y$$
$$\frac{\partial \sigma_{13}}{\partial x} + \frac{\partial \sigma_{23}}{\partial y} + \frac{\partial \sigma_{33}}{\partial z} + f_{V_z} = \rho \ddot{u}_z$$



Deformable solid and boundary conditions

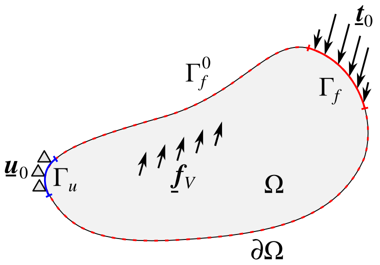
Notations:

- Consider a solid Ω with boundary $\partial\Omega$
- Boundary is split into Γ_u and Γ_f: $\partial\Omega = \Gamma_u \cup \Gamma_f$
- At Γ_u displacements $\underline{u}_0(t, \underline{X})$ are prescribed (Dirichlet boundary conditions [BC]):

  $\underline{u} = \underline{u}_0$ at Γ_u
- At Γ_f tractions $\underline{t}_0(t, \underline{X})$ are prescribed (Neumann BC):

  $\underline{\underline{\sigma}} \cdot \underline{n} = \underline{t}_0$ at Γ_f

  $\underline{\underline{\sigma}} \cdot \underline{n} = 0$ at Γ_f^0



Deformable solid and boundary conditions

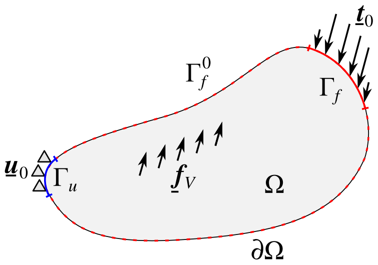
Notations:

- Consider a solid Ω with boundary $\partial\Omega$
- Boundary is split into Γ_u and Γ_f: $\partial\Omega = \Gamma_u \cup \Gamma_f$
- At Γ_u displacements $\underline{u}_0(t, \underline{X})$ are prescribed (Dirichlet boundary conditions [BC]):

  $\underline{u} = \underline{u}_0$ at Γ_u
- At Γ_f tractions $\underline{t}_0(t, \underline{X})$ are prescribed (Neumann BC):

  $\underline{\underline{\sigma}} \cdot \underline{n} = \underline{t}_0$ at Γ_f

  $\underline{\underline{\sigma}} \cdot \underline{n} = 0$ at Γ_f^0



Remarks:

- on the same boundary both BCs can be prescribed if they are orthogonal one to each other, i.e. $\underline{u}_0 \cdot \underline{t}_0 = 0$ (*ex.:* friction);
- a relationship between these BCs can be prescribed (Robin BC): $\underline{u}_0 = \underline{U} - k\underline{t}_0$ (*ex.:* Winkler's foundation).

Elastic and quasistatic problem set-up

- Equilibrium in absence of inertial forces

$$\nabla \cdot \underline{\underline{\sigma}} + \underline{f}_V = 0 \quad (*)$$

- Consistutive relation:

$$\underline{\underline{\sigma}} = {}^4\underline{\underline{C}} : \underline{\underline{\varepsilon}}$$

- Strain tensor:

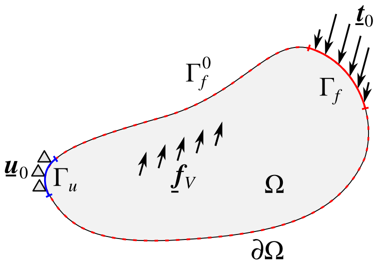
$$\underline{\underline{\varepsilon}} = \frac{1}{2}\left(\nabla \underline{u} + (\nabla \underline{u})^{\intercal}\right)$$

- Boundary conditions:

$\underline{u} = \underline{u}_0$ at Γ_u

$\underline{\underline{\sigma}} \cdot \underline{n} = \underline{t}_0$ at Γ_f

$\underline{\underline{\sigma}} \cdot \underline{n} = 0$ at Γ_f^0



- **Problem:**
find such field $\underline{u}$ in Ω that satisfies equilibrium Eq. (*) and boundary conditions.

Finite Element Method

Main idea

- From continuous to discrete problem
- Split solid into finite elements

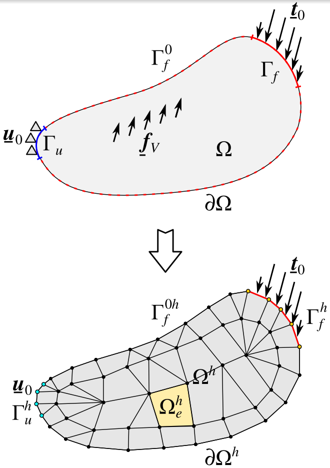
  $\Omega \to \Omega^h$ with $\Omega^h = \sum\limits_e \Omega^h_e$
- All quantities are associated with this discretization:

  $\underline{u} \to \underline{u}^h$, $\underline{\underline{\sigma}} \to \underline{\underline{\sigma}}^h$, $\Gamma_f \to \Gamma^h_f$, $\underline{t}_0 \to \underline{t}^h_0, \ldots$
- Search for $\underline{u}^h$ only in a finite number of points (nodes)
- Interpolate in between (within elements)
- Ensure (1) equilibrium of every element and (2) satisfaction of boundary conditions

  (1) $\nabla \cdot \underline{\underline{\sigma}}^h + \underline{f}^h_v = 0$ in $\Omega^h_e, \forall e$

  (2.a) $\underline{\underline{\sigma}}^h \cdot \underline{n}^h = \underline{t}^h_0$ at Γ^h_f

  (2.b) $\underline{u}^h = \underline{u}^h_0$ at Γ^h_u



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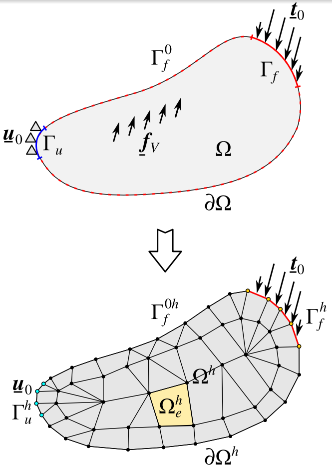
  $$\begin{aligned} &(1) \quad \nabla \cdot \underline{\underline{\sigma}}^h + \underline{f}^h_v = 0 \text{ in } \Omega^h_e, \forall e \\ &(2.a) \quad \underline{\underline{\sigma}}^h \cdot \underline{n}^h = \underline{t}^h_0 \text{ at } \Gamma^h_f \\ &(2.b) \quad \underline{u}^h = \underline{u}^h_0 \text{ at } \Gamma^h_u \end{aligned}$$



- Existence and uniqueness of the solution $\underline{u}^h_*$
- When discretization-size tends to zero $h \to 0$, convergence to the solution of the continuum problem: $\underline{u}^h_* \xrightarrow[h\to 0]{} \underline{u}_*$

Standard discrete system

1. For any discrete system the quantities of interest [q] depend on system parameters [p] and on locally acting external parameters [e]

$$[\mathbf{q}]_i = [\mathbf{q}]_i\left([\mathbf{p}]_j, [\mathbf{e}]_i\right)$$

2. In the first approximation this dependence is linear

$$\begin{aligned}
q_1 &= K_{11}p_1 + K_{12}p_2 + \dots K_{1N}p_N + A_{11}e_1 \\
q_2 &= K_{21}p_1 + K_{22}p_2 + \dots K_{2N}p_N + A_{22}e_2 \\
\dots \\
q_N &= K_{21}p_1 + K_{22}p_2 + \dots K_{2N}p_N + A_{NN}e_N
\end{aligned}$$

3. In matrix form

$$[\mathbf{q}]_i = [\mathbf{K}]_{ij}\,[\mathbf{p}]_j + [\mathbf{A}]_{ii}\,[\mathbf{e}]_i$$

4. Assuming that external parameters are of the same nature as quantities of interest ($[\mathbf{A}]_{ij} = [\mathbf{I}]_{ij}$)

$$\boxed{[\mathbf{q}]_i = [\mathbf{K}]_{ij}\,[\mathbf{p}]_j + [\mathbf{e}]_i}$$

Discrete system in structural mechanics

Main quantities

- Quantities of interest $[\mathbf{q}]$ are, in general, forces $[\mathbf{f}]$
- System parameters $[\mathbf{p}]$ are, in general, displacements $[\mathbf{u}]$
- External parameters $[\mathbf{e}]$ are, in general, external forces $[\mathbf{f}]^{ext}$

Main steps

1. Construct *stiffness matrix* and *nodal loads* vector
$$[\mathbf{K}]^k_{ij}\,, [\mathbf{f}]^k_i, \quad i,j \in 1, NN^k; k \in NE,$$
where NN^k is the number of nodes of k-th element, NE is the number of elements.
2. Assemble them into the global stiffness matrix and global load vector
$$[\mathbf{K}]_{ij}\,, [\mathbf{f}]_i, \quad i,j \in 1, NN,$$
where NN is the total number of nodes.
3. Add boundary conditions (for example Dirichlet and Neumann)
$$[\mathbf{f}]^{ext}_k, \quad k \in BC_f; \qquad [\mathbf{u}]^0_l, \quad l \in BC_u$$
4. Solve linear system of equations
$$[\mathbf{K}]_{ij}\,[\mathbf{u}]_j = [\mathbf{f}]_i - [\mathbf{f}]^{ext}_i \quad \to \quad [\mathbf{u}]_{j*}$$

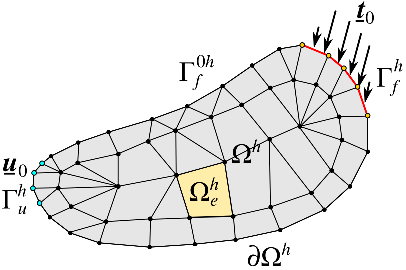
- Displacements are known at nodes: $\underline{\boldsymbol{u}}_i^h,\ i = 1,4$
- We need to know them inside the element
- Parametrize the inside with parameters $\{\xi,\eta\} \in [-1,1]$
- Use *interpolation* or *shape* functions $N_i(\xi,\eta)$ for position $\underline{\boldsymbol{X}}$

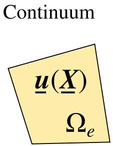
  $$\underline{\boldsymbol{X}}^h(\xi,\eta) = \sum_i \underline{\boldsymbol{X}}_i^h N_i(\xi,\eta)$$

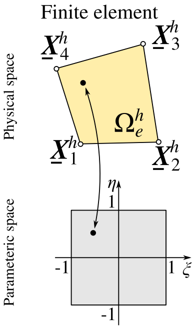
  and displacement $\underline{\boldsymbol{u}}$:

  $$\underline{\boldsymbol{u}}^h(\xi,\eta) = \sum_i \underline{\boldsymbol{u}}_i^h N_i(\xi,\eta)$$

- Remark: Find $\{\xi,\eta\}$ from $\underline{\boldsymbol{X}}$ is not always straigthforward (may result in a system of non-linear equations)







Shape functions

- Displacements are known at nodes: $\underline{\boldsymbol{u}}_i^h,\ i = 1,4$
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- Parametrize the inside with parameters $\{\xi,\eta\} \in [-1,1]$
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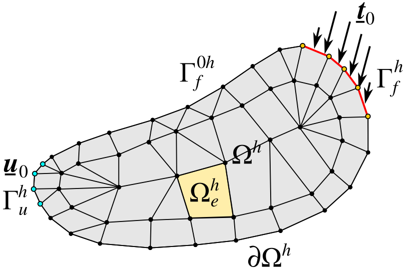
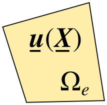
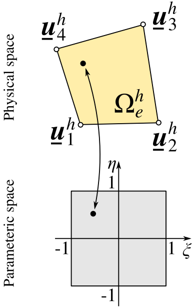


Continuum



Finite element



Shape functions II

Rules

- Node i has coordinates $\{\xi_i, \eta_i\}$
- Then $N_i(\xi_j, \eta_j) = \delta_{ij}$
- Partition of unity:

  $\forall \xi, \eta, :\ \sum\limits_i N_i(\xi, \eta) = 1$

Types

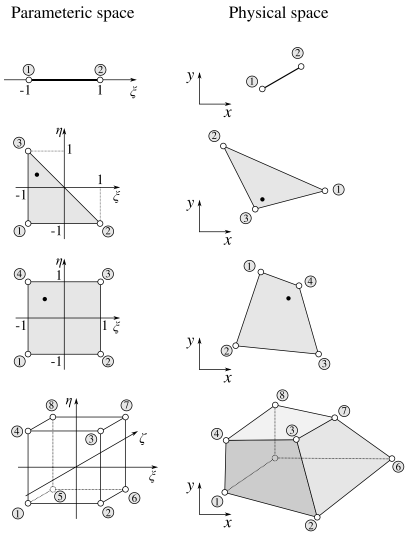
- Linear shape functions

  $$\frac{\partial N}{\partial \xi} = \text{const}$$

- Non-linear shape functions

  $$\frac{\partial N}{\partial \xi} = f(\xi)$$

- Linear elements vs quadratic elements
- Higher order elements



Shape functions III

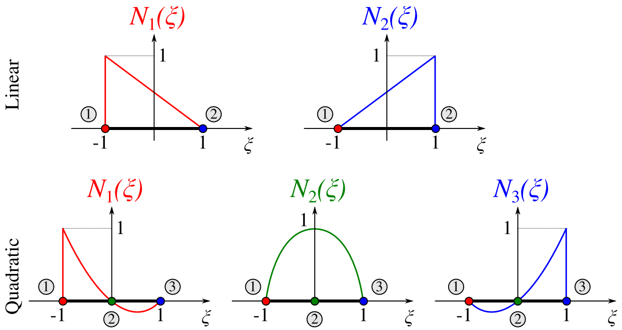
Example: bar element

- Linear shape functions:

$$N_1(\xi) = \frac{1}{2}(1-\xi)$$
$$N_2(\xi) = \frac{1}{2}(1+\xi)$$

- Quadratic shape functions:

$$N_1(\xi) = \frac{1}{2}\xi(\xi-1)$$
$$N_2(\xi) = (1-\xi^2)$$
$$N_3(\xi) = \frac{1}{2}\xi(1+\xi)$$



Shape functions: vectors and matrices

- Displacement nodal vectors $\underline{\boldsymbol{u}}_i = \underline{\boldsymbol{e}}_x u_i^x + \underline{\boldsymbol{e}}_y u_i^y$
- Array of nodal coordinates (size $\mathrm{dim} \cdot n$)
$$[\mathbf{X}] = [x_1,\ y_1,\ x_2,\ y_2,\ \dots\ x_n,\ y_n]^{\intercal}_{2n}$$
- Array of nodal displacements (size $\mathrm{dim} \cdot n$)
$$[\mathbf{u}] = [u_1^x,\ u_1^y,\ u_2^x,\ u_2^y,\ \dots\ u_n^x,\ u_n^y]^{\intercal}_{2n}$$
- Arrays of shape functions (size $\mathrm{dim} \cdot n$)
$$[\mathbf{N_x}] = [N_1,\ 0,\ N_2,\ 0,\ \dots\ N_n,\ 0]^{\intercal}_{2n}$$
$$[\mathbf{N_y}] = [0,\ N_1,\ 0,\ N_2,\ \dots\ 0,\ N_n]^{\intercal}_{2n}$$
$$[\mathbf{N}] = \begin{bmatrix} N_1 & 0 & N_2 & 0 & \dots & N_n & 0 \\ 0 & N_1 & 0 & N_2 & \dots & 0 & N_n \end{bmatrix}^{\intercal}_{2n\times\mathrm{dim}}$$
- Then
$$x(\xi,\eta,t) = [\mathbf{N_x}(\xi,\eta)]^{\intercal}[\mathbf{X}(\mathbf{t})], \quad y(\xi,\eta,t) = [\mathbf{N_y}(\xi,\eta)]^{\intercal}[\mathbf{X}(\mathbf{t})]$$
$$u^x(\xi,\eta,t) = [\mathbf{N_x}(\xi,\eta)]^{\intercal}[\mathbf{u}(\mathbf{t})], \quad u^y(\xi,\eta,t) = [\mathbf{N_y}(\xi,\eta)]^{\intercal}[\mathbf{u}(\mathbf{t})]$$

Gradients and shape functions

- Need to evaluate gradients (spatial derivatives) like $\frac{\partial f}{\partial x}$
- But with shape functions $f = f(\xi, \eta)$
- Then $\dfrac{\partial f(\xi,\eta)}{\partial x} = \dfrac{\partial f}{\partial \xi}\dfrac{\partial \xi}{\partial x} + \dfrac{\partial f}{\partial \eta}\dfrac{\partial \eta}{\partial x}$
- However, in general we do not have $\xi = \xi(x, y)$ but rather $x = x(\xi, \eta)$
- Let's do it other way around

$$\begin{bmatrix} \dfrac{\partial}{\partial \xi} \\ \dfrac{\partial}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial}{\partial x}\dfrac{\partial x}{\partial \xi} + \dfrac{\partial}{\partial y}\dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial}{\partial x}\dfrac{\partial x}{\partial \eta} + \dfrac{\partial}{\partial y}\dfrac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \begin{bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{bmatrix} = [\,\mathbf{J}\,] \begin{bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{bmatrix}$$

- Matrix $[\,\mathbf{J}\,]$ is called Jacobian operator and enables to obtain

$$\begin{bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{bmatrix} = [\,\mathbf{J}\,]^{-1} \begin{bmatrix} \dfrac{\partial}{\partial \xi} \\ \dfrac{\partial}{\partial \eta} \end{bmatrix}$$

Jacobian operator

- Jacobian operator or simply Jacobian:

$$[\mathbf{J}] = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}$$

- Using $x = [\mathbf{N_x}]^\intercal [\mathbf{X}], \quad y = [\mathbf{N_y}]^\intercal [\mathbf{X}]$ we get:

$$[\mathbf{J}] = \begin{bmatrix} [\mathbf{N}_{\mathbf{x},\xi}]^\intercal [\mathbf{X}] & [\mathbf{N}_{\mathbf{y},\xi}]^\intercal [\mathbf{X}] \\ [\mathbf{N}_{\mathbf{x},\eta}]^\intercal [\mathbf{X}] & [\mathbf{N}_{\mathbf{y},\eta}]^\intercal [\mathbf{X}] \end{bmatrix},$$

where $[\mathbf{N}_{\mathbf{x},\xi}] = \left[\frac{\partial N_1}{\partial \xi},\ 0,\ \frac{\partial N_2}{\partial \xi}, 0,\ \dots \frac{\partial N_n}{\partial \xi}, 0\right]^\intercal$ etc.

- Then the inverse Jacobian is given by:

$$[\mathbf{J}]^{-1} = \frac{1}{\Delta} \begin{bmatrix} [\mathbf{N}_{\mathbf{y},\eta}]^\intercal [\mathbf{X}] & -[\mathbf{N}_{\mathbf{y},\xi}]^\intercal [\mathbf{X}] \\ -[\mathbf{N}_{\mathbf{x},\eta}]^\intercal [\mathbf{X}] & [\mathbf{N}_{\mathbf{x},\xi}]^\intercal [\mathbf{X}] \end{bmatrix},$$

with $\Delta = \det([\mathbf{J}]) = [\mathbf{X}]^\intercal \left([\mathbf{N}_{\mathbf{x},\xi}][\mathbf{N}_{\mathbf{y},\eta}]^\intercal - [\mathbf{N}_{\mathbf{y},\xi}][\mathbf{N}_{\mathbf{x},\eta}]^\intercal\right)[\mathbf{X}] \neq 0$

Infinitesimal strain in 2D

- Strain tensor: $\underline{\underline{\varepsilon}} = \frac{1}{2}\left(\nabla\underline{u} + (\nabla\underline{u})^\intercal\right) \qquad (*)$
- Interpolated displacements: $u^x = [\mathbf{N_x}]^\intercal[\mathbf{u}], \quad u^y = [\mathbf{N_y}]^\intercal[\mathbf{u}]$
- Displacement gradient:

$$\nabla\underline{u} = \underline{e}_x \otimes \frac{\partial\underline{u}^h}{\partial x} + \underline{e}_y \otimes \frac{\partial\underline{u}^h}{\partial y} = \underline{e}^x \otimes \underline{e}^x \frac{\partial u^x}{\partial x} + \underline{e}^x \otimes \underline{e}^y \frac{\partial u^y}{\partial x} + \underline{e}^y \otimes \underline{e}^x \frac{\partial u^x}{\partial y} + \underline{e}^y \otimes \underline{e}^y \frac{\partial u^y}{\partial y}$$

$$\nabla\underline{u} \sim \begin{bmatrix} \frac{\partial u^x}{\partial x} & \frac{\partial u^y}{\partial x} \\ \frac{\partial u^x}{\partial y} & \frac{\partial u^y}{\partial y} \end{bmatrix} = [\,\mathbf{J}\,]^{-1} \begin{bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{bmatrix} \begin{bmatrix} u^x \\ u^y \end{bmatrix}^\intercal = [\,\mathbf{J}\,]^{-1} \begin{bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{bmatrix} \begin{bmatrix} [\mathbf{N_x}]^\intercal[\mathbf{u}] \\ [\mathbf{N_y}]^\intercal[\mathbf{u}] \end{bmatrix}^\intercal$$

- Finally $[\mathbf{E}] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}^\intercal [\,\mathbf{J}\,]^{-1} \begin{bmatrix} \partial[\mathbf{N_x}]^\intercal/\partial\xi & \partial[\mathbf{N_y}]^\intercal/\partial\xi \\ \partial[\mathbf{N_x}]^\intercal/\partial\eta & \partial[\mathbf{N_y}]^\intercal/\partial\eta \end{bmatrix} [\mathbf{u}]$

$$\varepsilon_{xx} = \left([\,\mathbf{J}\,]^{-1}_{11}[\mathbf{N}_{\mathbf{x},\xi}] + [\,\mathbf{J}\,]^{-1}_{12}[\mathbf{N}_{\mathbf{x},\eta}]\right)^\intercal[\mathbf{u}] = \frac{1}{\Delta}\left([\mathbf{N}_{\mathbf{y},\eta}]^\intercal[\mathbf{X}][\mathbf{N}_{\mathbf{x},\xi}] - [\mathbf{N}_{\mathbf{y},\xi}]^\intercal[\mathbf{X}][\mathbf{N}_{\mathbf{x},\eta}]\right)^\intercal[\mathbf{u}]$$

$$\varepsilon_{yy} = \left([\,\mathbf{J}\,]^{-1}_{21}[\mathbf{N}_{\mathbf{y},\xi}] + [\,\mathbf{J}\,]^{-1}_{22}[\mathbf{N}_{\mathbf{y},\eta}]\right)^\intercal[\mathbf{u}] = \frac{1}{\Delta}\left(-[\mathbf{N}_{\mathbf{x},\eta}]^\intercal[\mathbf{X}][\mathbf{N}_{\mathbf{y},\xi}] + [\mathbf{N}_{\mathbf{x},\xi}]^\intercal[\mathbf{X}][\mathbf{N}_{\mathbf{y},\eta}]\right)^\intercal[\mathbf{u}]$$

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u^x}{\partial y} + \frac{\partial u^y}{\partial x}\right) = \frac{1}{2}\left([\,\mathbf{J}\,]^{-1}_{11}[\mathbf{N}_{\mathbf{y},\xi}] + [\,\mathbf{J}\,]^{-1}_{12}[\mathbf{N}_{\mathbf{y},\eta}] + [\,\mathbf{J}\,]^{-1}_{21}[\mathbf{N}_{\mathbf{x},\xi}] + [\,\mathbf{J}\,]^{-1}_{22}[\mathbf{N}_{\mathbf{x},\eta}]\right)^\intercal[\mathbf{u}]$$

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$$\nabla\underline{u} \sim \begin{bmatrix} \dfrac{\partial u^x}{\partial x} & \dfrac{\partial u^y}{\partial x} \\ \dfrac{\partial u^x}{\partial y} & \dfrac{\partial u^y}{\partial y} \end{bmatrix} = [\,\mathbf{J}\,]^{-1} \begin{bmatrix} \dfrac{\partial}{\partial \xi} \\ \dfrac{\partial}{\partial \eta} \end{bmatrix} \begin{bmatrix} u^x \\ u^y \end{bmatrix}^\intercal = [\,\mathbf{J}\,]^{-1} \begin{bmatrix} \dfrac{\partial}{\partial \xi} \\ \dfrac{\partial}{\partial \eta} \end{bmatrix} \begin{bmatrix} [\mathbf{N_x}]^\intercal[\mathbf{u}] \\ [\mathbf{N_y}]^\intercal[\mathbf{u}] \end{bmatrix}^\intercal$$

- Finally $[\mathbf{E}] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}^\intercal [\,\mathbf{J}\,]^{-1} \begin{bmatrix} \partial[\mathbf{N_x}]^\intercal/\partial\xi & \partial[\mathbf{N_y}]^\intercal/\partial\xi \\ \partial[\mathbf{N_x}]^\intercal/\partial\eta & \partial[\mathbf{N_y}]^\intercal/\partial\eta \end{bmatrix} [\mathbf{u}]$

$$\varepsilon_{xx} = \left([\,\mathbf{J}\,]^{-1}_{11}[\mathbf{N}_{\mathbf{x},\xi}] + [\,\mathbf{J}\,]^{-1}_{12}[\mathbf{N}_{\mathbf{x},\eta}]\right)^\intercal[\mathbf{u}] = \frac{1}{\Delta}\left([\mathbf{N}_{\mathbf{y},\eta}]^\intercal[\mathbf{X}][\mathbf{N}_{\mathbf{x},\xi}] - [\mathbf{N}_{\mathbf{y},\xi}]^\intercal[\mathbf{X}][\mathbf{N}_{\mathbf{x},\eta}]\right)^\intercal[\mathbf{u}]$$

$$\varepsilon_{yy} = \left([\,\mathbf{J}\,]^{-1}_{21}[\mathbf{N}_{\mathbf{y},\xi}] + [\,\mathbf{J}\,]^{-1}_{22}[\mathbf{N}_{\mathbf{y},\eta}]\right)^\intercal[\mathbf{u}] = \frac{1}{\Delta}\left(-[\mathbf{N}_{\mathbf{x},\eta}]^\intercal[\mathbf{X}][\mathbf{N}_{\mathbf{y},\xi}] + [\mathbf{N}_{\mathbf{x},\xi}]^\intercal[\mathbf{X}][\mathbf{N}_{\mathbf{y},\eta}]\right)^\intercal[\mathbf{u}]$$

$$\varepsilon_{xy} = \frac{1}{2\Delta}\left([\mathbf{N}_{\mathbf{y},\eta}]^\intercal[\mathbf{X}][\mathbf{N}_{\mathbf{y},\xi}] - [\mathbf{N}_{\mathbf{y},\xi}]^\intercal[\mathbf{X}][\mathbf{N}_{\mathbf{y},\eta}] - [\mathbf{N}_{\mathbf{x},\eta}]^\intercal[\mathbf{X}][\mathbf{N}_{\mathbf{x},\xi}] + [\mathbf{N}_{\mathbf{x},\xi}]^\intercal[\mathbf{X}][\mathbf{N}_{\mathbf{x},\eta}]\right)^\intercal[\mathbf{u}]$$

Infinitesimal strain in 2D in matrix form

- Strain tensor: $\underline{\underline{\varepsilon}} = \frac{1}{2}\left(\nabla \underline{u} + (\nabla \underline{u})^\intercal\right) \qquad (*)$
- Represent it as an array (Voigt notations):

$$\underline{\underline{\varepsilon}} \quad \Rightarrow \quad [\mathbf{E}] = \left[\varepsilon_{xx}, \quad \varepsilon_{yy}, \quad \gamma_{xy}\right]^\intercal, \quad \gamma_{xy} = 2\varepsilon_{xy}$$

- Then

$$[\mathbf{E}]_3 = [\mathbf{B}]^\intercal_{3\times 2n}[\mathbf{u}]_{2n}$$

- With $[\mathbf{B}]$ given by:

$$[\mathbf{B}]^\intercal = \frac{1}{\Delta}\begin{bmatrix} \left([\mathbf{N}_{\mathbf{y},\eta}]^\intercal[\mathbf{X}][\mathbf{N}_{\mathbf{x},\xi}] - [\mathbf{N}_{\mathbf{y},\xi}]^\intercal[\mathbf{X}][\mathbf{N}_{\mathbf{x},\eta}]\right)^\intercal \\ \left(-[\mathbf{N}_{\mathbf{x},\eta}]^\intercal[\mathbf{X}][\mathbf{N}_{\mathbf{y},\xi}] + [\mathbf{N}_{\mathbf{x},\xi}]^\intercal[\mathbf{X}][\mathbf{N}_{\mathbf{y},\eta}]\right)^\intercal \\ \left([\mathbf{N}_{\mathbf{y},\eta}]^\intercal[\mathbf{X}][\mathbf{N}_{\mathbf{y},\xi}] - [\mathbf{N}_{\mathbf{y},\xi}]^\intercal[\mathbf{X}][\mathbf{N}_{\mathbf{y},\eta}] - [\mathbf{N}_{\mathbf{x},\eta}]^\intercal[\mathbf{X}][\mathbf{N}_{\mathbf{x},\xi}] + [\mathbf{N}_{\mathbf{x},\xi}]^\intercal[\mathbf{X}][\mathbf{N}_{\mathbf{x},\eta}]\right)^\intercal \end{bmatrix}_{3\times 2n}$$

Infinitesimal strain in 2D: example

- Consider a linear triangular element with shape functions:

$$N_1 = -\tfrac{1}{2}(\xi + \eta), \quad N_2 = \tfrac{1}{2}(1 + \xi), \quad N_3 = \tfrac{1}{2}(1 + \eta)$$

- Their derivatives are given by:

$$N_{1,\xi} = -1/2, \quad N_{2,\xi} = 1/2, \quad N_{3,\xi} = 0$$

$$N_{1,\eta} = -1/2, \quad N_{2,\eta} = 0, \quad N_{3,\eta} = 1/2$$

$$\Delta = \frac{1}{4}((x_2 - x_1)(y_3 - y_1) - (y_2 - y_1)(x_3 - x_1))^{*}$$

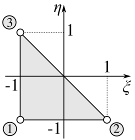
- Then

$$\varepsilon_{xx} = \frac{1}{4\Delta}\left[(y_3 - y_1)(u^x_2 - u^x_1) - (y_2 - y_1)(u^x_3 - u^x_1)\right]$$

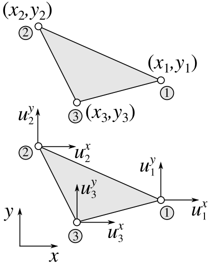
$$\varepsilon_{yy} = \frac{1}{4\Delta}\left[(x_2 - x_1)(u^y_3 - u^y_1) - (x_3 - x_1)(u^y_2 - u^y_1)\right]$$

$$\gamma_{xy} = \frac{1}{4\Delta}\left[(y_3 - y_1)(u^y_2 - u^y_1) - (y_2 - y_1)(u^y_3 - u^y_1) + (x_2 - x_1)(u^x_3 - u^x_1) - (x_3 - x_1)(u^x_2 - u^x_1)\right]$$

Parametric space



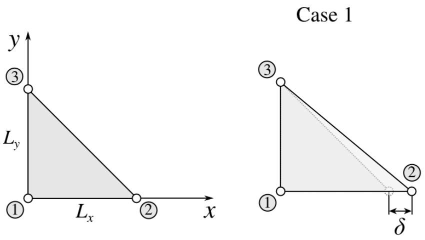
Physical space



*Half of the area of the triangle.

Infinitesimal strain in 2D: example II

- Rectangular triangle $x_1 = x_3,\ y_1 = y_2,\ \Delta = L_x L_y/4$
- Case 1: pure tension/compression along OX iaoi $u_3^y = u_1^y, u_2^y = u_1^y, u_3^x = u_1^x$
  Ex.: $u_2^x = \delta :\quad \varepsilon_{xx} = \frac{1}{4\Delta}(y_3 - y_1)(u_2^x - u_1^x) = \delta/L_x,\quad \varepsilon_{yy} = \gamma_{xy} = 0$

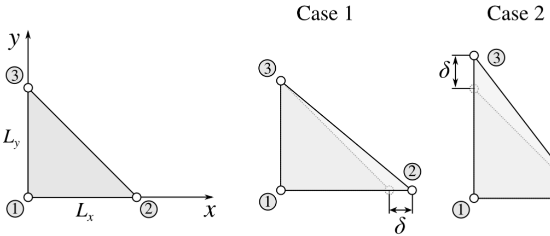


Reference configuration

Current configuration

Infinitesimal strain in 2D: example II

- Rectangular triangle $x_1 = x_3,\; y_1 = y_2,\; \Delta = L_x L_y/4$
- Case 1: pure tension/compression along OX iaoi $u_3^y = u_1^y,\, u_2^y = u_1^y,\, u_3^x = u_1^x$
  Ex.: $u_2^x = \delta : \quad \varepsilon_{xx} = \frac{1}{4\Delta}(y_3 - y_1)(u_2^x - u_1^x) = \delta/L_x, \quad \varepsilon_{yy} = \gamma_{xy} = 0$
- Case 2: pure tension/compression along OY iaoi $u_2^x = u_1^x,\, u_2^y = u_1^y,\, u_3^x = u_1^x$
  Ex.: $u_3^y = \delta : \quad \varepsilon_{yy} = \frac{1}{4\Delta}(x_2 - x_1)(u_3^y - u_1^y) = \delta/L_y, \quad \varepsilon_{xx} = \gamma_{xy} = 0$



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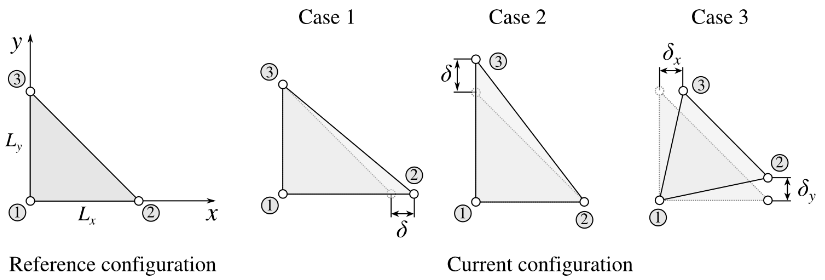
  Ex.: $u_3^y = \delta: \quad \varepsilon_{yy} = \frac{1}{4\Delta}(x_2 - x_1)(u_3^y - u_1^y) = \delta/L_y, \quad \varepsilon_{xx} = \gamma_{xy} = 0$
- Case 3: pure shear in XY iaoi $u_2^x = u_1^x,\ u_3^y = u_1^y$
  Ex.: $u_2^y = \delta_y,\ u_3^x = \delta_x:$
  $$\gamma_{xy} = \frac{1}{4\Delta}\Big((y_3 - y_1)(u_2^y - u_1^y) + (x_2 - x_1)(u_3^x - u_1^x)\Big) = \frac{\delta_y}{L_x} + \frac{\delta_x}{L_y}, \quad \varepsilon_{xx} = \varepsilon_{yy} = 0$$



Stress tensor

- In linear elasticity:

$$\underline{\underline{\sigma}} = {}^4\underline{\underline{C}} : (\underline{\underline{\varepsilon}} - \underline{\underline{\varepsilon}}_0) + \underline{\underline{\sigma}}_0$$

- Residual stress field $\underline{\underline{\sigma}}_0$
- Initial strain field $\underline{\underline{\varepsilon}}_0$
- In self equilibrated system: $\underline{\underline{\sigma}}_0 = {}^4\underline{\underline{C}} : \underline{\underline{\varepsilon}}_0$ resulting in

$$\underline{\underline{\sigma}} = {}^4\underline{\underline{C}} : (\underline{\underline{\varepsilon}} - \underline{\underline{\varepsilon}}_{th})$$

- With thermal strain field $\underline{\underline{\varepsilon}}_{th}$:

$$\underline{\underline{\varepsilon}}_{th} = \alpha(T - T_0)\underline{\underline{I}},$$

whereα is the coefficient of thermal expansion (CTE), T and T_0 are the current and reference temperature fields, respectively.

Stress: 2D isotropic elasticity

- Recall stress/strain relationship:

$$\underline{\underline{\sigma}} = \frac{\nu E}{(1+\nu)(1-2\nu)} \mathrm{tr}(\underline{\underline{\varepsilon}})\underline{\underline{I}} + \frac{E}{1+\nu}\underline{\underline{\varepsilon}}$$

- Stress (in Voigt notations): $\underline{\underline{\sigma}} \quad \Rightarrow \quad [\mathbf{S}] = [\sigma_{xx}, \quad \sigma_{yy}, \quad \sigma_{xy}]^\intercal$
- In plane stress $\sigma_{zz} = 0$, $\varepsilon_{zz} = \frac{\nu}{\nu-1}(\varepsilon_{xx} + \varepsilon_{yy})$
- In plain strain $\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy})$, $\varepsilon_{zz} = 0$
- Stress/strain relationship: $[\mathbf{S}]_i = [\mathbf{D}]_{ij}\, [\mathbf{E}]_j$
- Matrix $[\mathbf{D}]$ in plane strain $\varepsilon_{zz} = \varepsilon_{xz} = \varepsilon_{yz} = 0$:

$$[\mathbf{D}]_{ij} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & (1-2\nu)/2^* \end{bmatrix}$$

- Matrix $[\mathbf{D}]$ in plane stress $\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0$, $\mathrm{tr}(\underline{\underline{\varepsilon}}) = \frac{1-2\nu}{1-\nu}(\varepsilon_{xx} + \varepsilon_{yy})$:

$$[\mathbf{D}]_{ij} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2^* \end{bmatrix}$$

*Factor $1/2$ appears because γ_{xy} was inserted in $[\mathbf{E}]$ instead of ε_{xy}.

Stress: general case

Voigt notations in 3D case

- Stress tensor: $\underline{\underline{\sigma}} \to [\mathbf{S}] = [\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{yz}, \sigma_{xz}]^\intercal$
- Strain tensor: $\underline{\underline{\varepsilon}} \to [\mathbf{E}] = [\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, \gamma_{xy}, \gamma_{yz}, \gamma_{xz}]^\intercal$
- Hooke's law: $[\mathbf{S}] = [\mathbf{D}]\,[\mathbf{E}]$
- Isotropic elasticity (two constants E, ν):

$$[\mathbf{D}]_{ij} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & (1-2\nu)/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & (1-2\nu)/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & (1-2\nu)/2 \end{bmatrix}$$

- Cubic elasticity (3 constants E, ν, μ):

$$[\mathbf{D}]_{ij} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{44} \end{bmatrix}$$

Voigt notations in 3D case

- Transversely isotropic elasticity (5 constants $E_1, E_2, \nu_1, \nu_2, \mu_1$):

$$[\mathbf{D}]_{ij} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & (C_{11}-C_{12})/2 \end{bmatrix}$$

- Orthotropic elasticity (9 constants $E_{xx}, E_{yy}, E_{zz}, \nu_{xy}, \nu_{yz}, \nu_{xz}, \mu_{xy}, \mu_{yz}, \mu_{xz}$):

$$[\mathbf{D}]_{ij} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix}$$

Strain/Stress: spherical part

Spherical part of a tensor $= \dfrac{1}{3}\mathrm{tr}(\underline{\underline{A}})\underline{\underline{I}}$

- If the strain tensor can be presented as $\underline{\underline{\varepsilon}} = \frac{1}{3}\mathrm{tr}(\underline{\underline{\varepsilon}})\underline{\underline{I}}$,
  then only volume change happens at this location $\Delta V/V_0 = \mathrm{tr}(\underline{\underline{\varepsilon}})$

$$\underline{\underline{\varepsilon}} \sim \begin{bmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon \end{bmatrix}$$

- If the stress tensor can be presented as $\underline{\underline{\sigma}} = \frac{1}{3}\mathrm{tr}(\underline{\underline{\sigma}})\underline{\underline{I}}$,
  then the stress state is pure hydrostatic compression under pressure $p = -\mathrm{tr}(\sigma)/3$

$$\underline{\underline{\sigma}} \sim \begin{bmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix}$$

Strain/Stress: deviatoric part

Deviatoric part of a tensor $= \underline{\underline{A}} - \dfrac{1}{3}\mathrm{tr}(\underline{\underline{A}})\underline{\underline{I}}$

- If the strain tensor does not have spherical part $\underline{\underline{\varepsilon}} = \underline{\underline{\varepsilon}} - \frac{1}{3}\mathrm{tr}(\underline{\underline{\varepsilon}})\underline{\underline{I}}$, then no volume change happens at this location $\Delta V/V_0 = 0$ only the shape changes, Ex.:

$$\underline{\underline{\varepsilon}} \sim \begin{bmatrix} \varepsilon & 0 & 0 \\ 0 & -0.5\varepsilon & 0 \\ 0 & 0 & -0.5\varepsilon \end{bmatrix}, \qquad \underline{\underline{\varepsilon}} \sim \begin{bmatrix} 0 & \varepsilon & 0 \\ \varepsilon & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- If the stress tensor is presented only by deviatoric part $\underline{\underline{\sigma}} = \underline{\underline{\sigma}} - \frac{1}{3}\mathrm{tr}(\underline{\underline{\sigma}})\underline{\underline{I}}$, then the stress state is pure shear:

$$\underline{\underline{\sigma}} \sim \begin{bmatrix} -\sigma & 0 & 0 \\ 0 & 2\sigma & 0 \\ 0 & 0 & -\sigma \end{bmatrix}, \qquad \underline{\underline{\sigma}} \sim \begin{bmatrix} 0 & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & 0 & 0 \\ \sigma_{xz} & 0 & 0 \end{bmatrix}$$

- In general both parts are present: $\underline{\underline{\varepsilon}} = \underline{\underline{e}} + \mathrm{tr}(\underline{\underline{\varepsilon}})\underline{\underline{I}}/3$, $\underline{\underline{\sigma}} = \underline{\underline{s}} + \mathrm{tr}(\underline{\underline{\sigma}})\underline{\underline{I}}/3$

Strain/Stress: elastic relationships

- Recall: $\underline{\underline{\varepsilon}} = \underline{\underline{e}} + \frac{\Delta V}{3V}\underline{\underline{I}}, \quad \underline{\underline{\sigma}} = \underline{\underline{s}} - p\underline{\underline{I}}$
- For deviatoric part in linear isotropic elasticity

$$\underline{\underline{s}} = \frac{E}{1+\nu}\underline{\underline{e}}, \quad \underline{\underline{s}} = 2\mu\underline{\underline{e}},$$

where $\mu = \dfrac{E}{2(1+\nu)}$ is called *shear modulus*.

- For spherical parts

$$\mathrm{tr}(\underline{\underline{\varepsilon}}) = \frac{1-2\nu}{E}\mathrm{tr}(\sigma) = -\frac{3(1-2\nu)}{E}p$$

then

$$-\frac{1}{V}\frac{dV}{dp} = \frac{3(1-2\nu)}{E} \quad \Leftrightarrow \quad -V\frac{dp}{dV} = \frac{E}{3(1-2\nu)} = K$$

where $K = \dfrac{E}{3(1-2\nu)}$ is called *bulk modulus*.

Stress and reactions: element's equilibrium II

- Work of nodal forces on *virtual* nodal displacements $= \dfrac{1}{2}\,\underline{\boldsymbol{f}}_i \cdot \delta\underline{\boldsymbol{u}}_i$
- Work density of distributed volumetric forces $= \dfrac{1}{2}\,\underline{\boldsymbol{f}}_V \cdot \delta\underline{\boldsymbol{u}}_V$
- Corresponding density of elastic energy $= \dfrac{1}{2}\,\underline{\underline{\boldsymbol{\sigma}}} : \delta\underline{\underline{\boldsymbol{\varepsilon}}}$

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- Corresponding density of elastic energy = $\frac{1}{2}\underline{\underline{\sigma}} : \delta\underline{\underline{\varepsilon}}$
- Stored elastic energy equals this work:

$$\int\limits_{V^e} \underline{\underline{\sigma}} : \underline{\underline{\varepsilon}}\, dV = \sum_i \underline{f}_i \cdot \underline{u}_i + \int V^e \underline{f}_V \cdot \delta\underline{u}\, dV$$

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- Elastic stress $\underline{\underline{\sigma}} = {}^4\underline{\underline{C}} : (\underline{\underline{\varepsilon}} - \underline{\underline{\varepsilon}}_{th}) \Rightarrow [\mathbf{S}] = [\mathbf{D}]\,([\mathbf{E}] - [\mathbf{E}_{\mathbf{th}}])$
- Strain $\underline{\underline{\varepsilon}} \sim [\mathbf{E}] = [\mathbf{B}]^\intercal [\mathbf{u}]$, vol. force density $\underline{f}_v \sim [\mathbf{f_v}] = [f_v^x, f_v^y, f_v^z]^\intercal$, volumetric virt. displacement $\delta\underline{u}_V \sim [\mathbf{N}]^\intercal \delta[\mathbf{u}]$:

$$\int\limits_{V^e} \left\{ ([\mathbf{D}]\,([\mathbf{E}] - [\mathbf{E}_{\mathbf{th}}]))^\intercal\, \delta[\mathbf{E}] - [\mathbf{f_v}]^\intercal [\mathbf{N_i}]^\intercal \delta[\mathbf{u}] \right\} dV = [\mathbf{f}]^\intercal \delta[\mathbf{u}]$$

Stress and reactions: element's equilibrium II

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$$\int\limits_{V^e} \left\{([\mathbf{D}]\,([\mathbf{E}] - [\mathbf{E}_{\mathbf{th}}]))^\intercal\, \delta[\mathbf{E}] - [\mathbf{f_v}]^\intercal [\mathbf{N_i}]^\intercal \delta[\mathbf{u}]\right\} dV = [\mathbf{f}]^\intercal \delta[\mathbf{u}]$$

$$[\mathbf{u}]\left[\int\limits_{V^e} [\mathbf{B}]\,[\mathbf{D}]\,[\mathbf{B}]^\intercal\, dV\right]\delta[\mathbf{u}] - \left[\int\limits_{V^e}\left([\mathbf{f_v}]^\intercal [\mathbf{N_i}]^\intercal + [\mathbf{E}_{\mathbf{th}}]^\intercal\, [\mathbf{D}]\,[\mathbf{B}]^\intercal\right) dV\right]\delta[\mathbf{u}] = [\mathbf{f}]^\intercal \delta[\mathbf{u}]$$

Stress and reactions: element's equilibrium II

- Balance of virtual work for a single element:

$$[\mathbf{u}]\left[\int\limits_{V^e}[\mathbf{B}]\,[\mathbf{D}]\,[\mathbf{B}]^\intercal\,dV\right]\delta[\mathbf{u}] - \left[\int\limits_{V^e}\Big([\mathbf{f_v}]^\intercal[\mathbf{N_i}]^\intercal + [\mathbf{E_{th}}]^\intercal\,[\mathbf{D}]\,[\mathbf{B}]^\intercal\Big)\,dV\right]\delta[\mathbf{u}] = [\mathbf{f}]^\intercal\,\delta[\mathbf{u}]$$

- For arbitrary virtual displacements $\delta[\mathbf{u}]$:

$$\underbrace{\left[\int\limits_{V^e}[\mathbf{B}]^\intercal\,[\mathbf{D}]\,[\mathbf{B}]\,dV\right]}_{[\mathbf{K^e}]}[\mathbf{u}] + \underbrace{\left[\int\limits_{V^e}\Big(-[\mathbf{f_v}]^\intercal[\mathbf{N_i}] - [\mathbf{B}]\,[\mathbf{D}]\,[\mathbf{E_{th}}]\Big)\,dV\right]}_{[\mathbf{f^e_{int}}]} = \underbrace{[\mathbf{f}]}_{[\mathbf{f^e_{ext}}]}$$

- System of equations linking displacements and reactions:

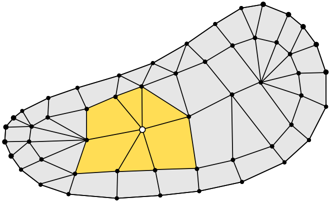
$$\boxed{[\mathbf{K^e}]\,[\mathbf{u^e}] + [\mathbf{f^e_{int}}] = [\mathbf{f^e_{ext}}]}$$

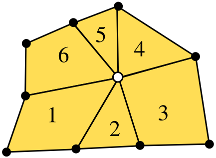
Assembly

- At every internal node the total force should be zero:

$$\sum_e [\mathbf{f}^{\mathbf{e}}_{\mathbf{ext}}] = 0$$

summation over all elements e attached to this node.





- Summation over all nodes gives:

$$[\mathbf{K}]\,[\mathbf{u}] + [\mathbf{f}_{\mathbf{int}}] = 0$$

Dirichlet boundary conditions

Dirichlet BC

- Use penalty method to enforce prescribed displacements: array $[\mathbf{u}_0] = [0\dots0\;u_{i0}\;0\dots0\;u_{j0}\;0]$
- Diagonal selection matrix $[\mathbf{I}^s]$ with ones at prescribed degrees of freedom (DOFs):

$$[\mathbf{I}^s] = \begin{array}{c} \begin{array}{ccccccccc} & & & \overbrace{}^{i} & & & & \overbrace{}^{j} & \end{array} \\ \left[\begin{array}{ccccccccc} 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & \dots & 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ \\ \\ \}\; i \\ \\ \\ \\ \}\; j \\ \\ \end{array} \end{array}$$

- Then the system is changed to

$$\left([\mathbf{K}] + \epsilon\,[\mathbf{I}^s]\right)[\mathbf{u}] = \left([\mathbf{I}] - [\mathbf{I}^s]\right)\left([\mathbf{f}_{\text{ext}}] - [\mathbf{f}_{\text{int}}]\right) + \epsilon[\mathbf{u}_0]$$

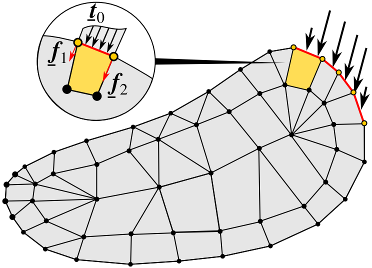
where ϵ is the penalty coefficient such that $\epsilon \gg \max(K_{ij})$, and $[\mathbf{I}]$ is the identity matrix.

Neumann boundary conditions

Neumann BC

- Surface traction $\underline{t}_0$ at Γ_f
- Virtual work of surface traction over one element:

$$\int\limits_{\Gamma_f^e} \underline{t}_0 \cdot \delta \underline{u} \, d\Gamma = \underline{f}_{ext}^i \cdot \delta \underline{u}_i^e$$



- Then

$$[\mathbf{f}_{\mathbf{ext}}^{\mathbf{i}}] = \int\limits_{\Gamma_f^e} [\mathbf{t}_0]^\intercal [\mathbf{N}]^\intercal \, d\Gamma$$

Discrete system of equations

- Balance of virtual work for the whole body:

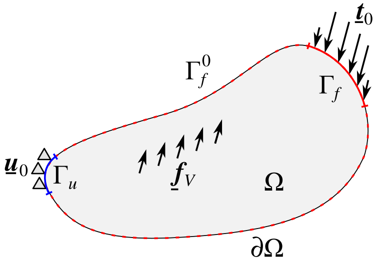
$$\underbrace{\left[\int\limits_V [\mathbf{B}]^\intercal\,[\mathbf{D}]\,[\mathbf{B}]\,dV\right]}_{[\mathbf{K}]}[\mathbf{u}] = \underbrace{\int\limits_{\Gamma_f} [\mathbf{t_0}]^\intercal [\mathbf{N}]^\intercal\,d\Gamma}_{[\mathbf{f}_{\text{ext}}]} + \underbrace{\left[\int\limits_V \Big([\mathbf{f_v}]^\intercal [\mathbf{N_i}] + [\mathbf{B}]\,[\mathbf{D}]\,[\mathbf{E}_{\,\text{th}}]\Big)\,dV\right]}_{-[\mathbf{f}_{\text{int}}]}$$

- System of equations linking displacements and reactions:

$$\boxed{[\mathbf{K}]\,[\mathbf{u}] = [\mathbf{f}_{\text{ext}}] - [\mathbf{f}_{\text{int}}]}$$

- Stiffness matrix $[\mathbf{K}]$
- Vector of degrees of freedom (DOFs) $[\mathbf{u}]$
- Right hand term (vector of forces) $[\mathbf{f}_{\text{ext}}] - [\mathbf{f}_{\text{int}}]$

Different approach: virtual work formulation I



- Arbitrary virtual displacements $\delta\underline{u}$
- Strong form: $\nabla\cdot\underline{\underline{\sigma}} + \underline{f}_V = 0$ + BCs
- Take a product with virtual displacements and integrate over Ω:

$$\int\limits_{\Omega} \left(\nabla\cdot\underline{\underline{\sigma}}\cdot\delta\underline{u} + \underline{f}_V\cdot\delta\underline{u}\right) dV = 0$$

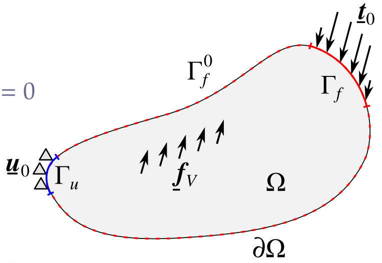
- Replacement: $\nabla\cdot\underline{\underline{\sigma}}\cdot\delta\underline{u} = \nabla\cdot\left(\underline{\underline{\sigma}}\cdot\delta\underline{u}\right) - \underline{\underline{\sigma}} : \nabla\delta\underline{u}$
- Following Gauss-Ostrogradsky theorem: $\int\limits_V \nabla\cdot(\bullet)\,dV = \int\limits_S \underline{n}\cdot(\bullet)\,dS$
- So

$$\int\limits_{\partial\Omega} \underline{n}\cdot\underline{\underline{\sigma}}\cdot\delta\underline{u}\,dS + \int\limits_{\Omega}\left(\underline{f}_V\cdot\delta\underline{u} - \underline{\underline{\sigma}} : \delta\underline{\underline{\varepsilon}}\right) dV = 0$$

continue. . .

- Weak form

$$\int\limits_{\partial\Omega} \underline{n}\cdot\underline{\underline{\sigma}}\cdot\delta\underline{u}\,dS + \int\limits_{\Omega}\left(\underline{f}_V\cdot\delta\underline{u} - \underline{\underline{\sigma}} : \delta\underline{\underline{\varepsilon}}\right)dV = 0$$



- Non-trivial Neumann boundary conditions at Γ_f

$$\int\limits_{\Omega}\underline{\underline{\sigma}} : \delta\underline{\underline{\varepsilon}}\,dV = \int\limits_{\Gamma_f}\underline{t}_0\cdot\delta\underline{u}\,dS + \int\limits_{\Omega}\underline{f}_V\cdot\delta\underline{u}\,dV$$

- **Remark I:** in the strong form $\underline{u}$ should be C^2-smooth, in the weak form $\underline{u}$ should be only square-integrable as well as its first derivative, thus $\underline{u}\in\mathbb{H}^1$, i.e. from Sobolev's functional space of the first order. In addition $\underline{u} = \underline{u}_0$ at Γ_u
- **Remark II:** for linear elasticity, the stress tensor* $\underline{\underline{\sigma}} = {}^4\underline{\underline{C}} : (\underline{\underline{\varepsilon}} - \underline{\underline{\varepsilon}}_{th})$

$$\int\limits_{\Omega}\underline{\underline{\varepsilon}} : {}^4\underline{\underline{C}} : \delta\underline{\underline{\varepsilon}}\,dV = \int\limits_{\Gamma_f}\underline{t}_0\cdot\delta\underline{u}\,dS + \int\limits_{\Omega}\left(\underline{f}_V + {}^4\underline{\underline{C}} : \underline{\underline{\varepsilon}}_{th}\right)\cdot\delta\underline{u}\,dV$$

Different approach II: potential energy

Remark III:

- If the system remains linear (boundary conditions, linear elasticity)
- The principle of virtual work is equivalent to the minimum of the total potential energy
- {Potential energy} = {Internal energy} - {Work of all forces}

$$\Pi(\underline{u}, \underline{t}_0, \underline{u}_0) = \frac{1}{2}\int\limits_{\Omega} \underline{\underline{\sigma}} : \underline{\underline{\varepsilon}} \, dV - \int\limits_{\Gamma_f} \underline{t}_0 \cdot \underline{u} \, d\Gamma - \int\limits_{\Omega} \underline{f}_V \cdot \underline{u} \, dV$$

- Stationary point of the total potential energy $\dfrac{\partial \Pi}{\partial \underline{u}} = 0$ for given loads $\underline{t}_0, \underline{u}_0$:

$$\frac{\partial \Pi}{\partial \underline{u}} = \int\limits_{\Omega} \underline{\underline{\varepsilon}} : {}^4\underline{\underline{C}} : \frac{\partial \underline{\underline{\varepsilon}}}{\partial \underline{u}} \, dV - \int\limits_{\Gamma_f} \underline{t}_0 \, d\Gamma - \int\limits_{\Omega} \underline{f}_V \, dV = 0$$

- The same equation

Evaluation of the integrals

- Weak form (recall):

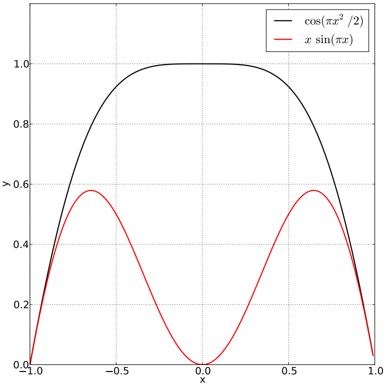
$$\underbrace{\left[\int\limits_V [\mathbf{B}]^\intercal\,[\mathbf{D}]\,[\mathbf{B}]\,dV\right]}_{[\mathbf{K}]}[\mathbf{u}] = \underbrace{\int\limits_{\Gamma_f} [\mathbf{t_0}]^\intercal\,[\mathbf{N}]^\intercal\,d\Gamma}_{[\mathbf{f_{ext}}]} + \underbrace{\left[\int\limits_V \Big(\,[\mathbf{f_v}]^\intercal\,[\mathbf{N_i}] + [\mathbf{B}]\,[\mathbf{D}]\,[\mathbf{E}_{\mathbf{th}}]\Big)\,dV\right]}_{-[\mathbf{f_{int}}]}$$

- Exact integration: $\int\limits_a^b f(x)dx = F(b) - F(a)$ (not always possible)
- Approximate integration (trapezoidal rule, Simpson's rule)
- Gauss quadrature: $\int\limits_a^b f(x)dx \approx \sum\limits_{i=1}^{N_{GP}} w_i f(x_i)$
- Gauss points x_i with $i = 1, N_{GP}$
- Integration is exact for polynomials of order $2N_{GP} - 1$
- Tabulated data for x_i, w_i (1D, 2D, 3D integration)

Evaluation of the integrals: example

- Function $f(x) = \cos(\pi x^2/2)$
  - $N_{GP} = 1$: error ≈ 28.22 %
  - $N_{GP} = 2$: error ≈ 11.04 %
  - $N_{GP} = 3$: error ≈ 1.14 %
  - $N_{GP} = 4$: error ≈ 0.14 %
  - $N_{GP} = 5$: error ≈ 0.01 %

- Function $f(x) = x\,\sin(\pi x)$
  - $N_{GP} = 1$: error ≈ 100.00 %
  - $N_{GP} = 2$: error ≈ 76.05 %
  - $N_{GP} = 3$: error ≈ 12.07 %
  - $N_{GP} = 4$: error ≈ 0.80 %
  - $N_{GP} = 5$: error ≈ 0.03 %



Evaluation of the integrals II

- Consider: $\int\limits_V [\mathbf{B}]^\intercal\,[\mathbf{D}]\,[\mathbf{B}]\,dV = \sum\limits_{e=1}^{N_e} \int\limits_{V_e} [\mathbf{B}]^\intercal\,[\mathbf{D}]\,[\mathbf{B}]\,dV$
- Transpose to the parametric space (2D example)

$$\int\limits_{V_e} [\mathbf{B}(\xi,\eta)]^\intercal\,[\mathbf{D}]\,[\mathbf{B}(\xi,\eta)]\,dV = \int\limits_{-1}^{1}\int\limits_{-1}^{1} [\mathbf{B}(\xi,\eta)]^\intercal\,[\mathbf{D}]\,[\mathbf{B}(\xi,\eta)]\,\mathrm{det}([\,\mathbf{J}\,])\,d\xi d\eta$$

- Finally:

$$[\mathbf{K}] = \int\limits_V [\mathbf{B}]^\intercal\,[\mathbf{D}]\,[\mathbf{B}]\,dV = \sum\limits_{e=1}^{N_e}\sum\limits_{GP=1}^{N_{GP}} [\mathbf{B^e}(\xi_{\mathbf{GP}},\eta_{\mathbf{GP}})]^\intercal\,[\mathbf{D}]\,[\mathbf{B^e}(\xi_{\mathbf{GP}},\eta_{\mathbf{GP}})]\,\mathrm{det}([\,J^e(xi_{GP},\eta_{GP})\,])w_{GP}$$

Evaluation of the integrals III

- If $N(\xi,\eta) = P_p$ is a polynomial of order p, then $[\,\mathbf{J}\,] = P_{\dim(p-1)}$, $[\mathbf{B}] = \frac{P_{2(p-1)}}{Q_{\dim(p-1)}}$
- **Remark I:** Gauss quadrature is exact for $p = 1$ and approximate if $p > 1$.
- **Remark II:** Stress and strains are exactly evaluated only in Gauss points, in all other points they are extrapolated/interpolated
- **Remark III:** 1 GP for linear triangle, 3 GP for quadratic triangle, 4 GP for bilinear quadrilateral element, 9 GP for quadratic quadrilateral, etc.
- **Remark IV:** Underintegration may lead to zero-energy deformation modes (which are often stabilized in FE software)

- Shape functions:

$$N_1 = \tfrac{1}{4}(1-\xi)(1-\eta), \quad N_2 = \tfrac{1}{4}(1+\xi)(1-\eta)$$
$$N_3 = \tfrac{1}{4}(1+\xi)(1+\eta), \quad N_4 = \tfrac{1}{4}(1-\xi)(1+\eta)$$

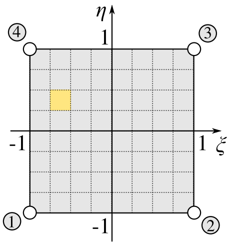
- Shape function derivatives:

$$N_{1,\xi} = -\tfrac{1}{4}(1-\eta), \quad N_{2,\xi} = \tfrac{1}{4}(1-\eta)$$
$$N_{3,\xi} = \tfrac{1}{4}(1+\eta), \quad N_{4,\xi} = -\tfrac{1}{4}(1+\eta)$$
$$N_{1,\eta} = -\tfrac{1}{4}(1-\xi), \quad N_{2,\eta} = -\tfrac{1}{4}(1+\xi)$$
$$N_{3,\eta} = \tfrac{1}{4}(1+\xi), \quad N_{4,\eta} = \tfrac{1}{4}(1-\xi)$$

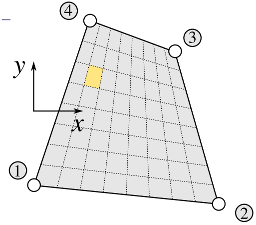
- Determinant of Jacobian ($dA = \det[\,\mathbf{J}\,]d\xi d\eta$):

$$\det([\,\mathbf{J}\,]) =$$
$$\tfrac{1}{16}\left[((1-\eta)(x_2-x_1)+(1+\eta)(x_3-x_4))((1+\xi)(y_3-y_2)+(1-\xi)(y_4-y_1)) - \right.$$
$$\left. -\ ((1-\eta)(y_2-y_1)+(1+\eta)(y_3-y_4))((1+\xi)(x_3-x_2)+(1-\xi)(x_4-x_1))\right]$$

Parameteric space



Physical space



- Shape functions:

$$N_1 = \tfrac14(1-\xi)(1-\eta), \quad N_2 = \tfrac14(1+\xi)(1-\eta)$$
$$N_3 = \tfrac14(1+\xi)(1+\eta), \quad N_4 = \tfrac14(1-\xi)(1+\eta)$$

- Shape function derivatives:

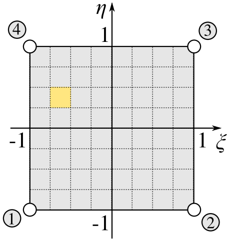
$$N_{1,\xi} = -\tfrac14(1-\eta), \quad N_{2,\xi} = \tfrac14(1-\eta)$$
$$N_{3,\xi} = \tfrac14(1+\eta), \quad N_{4,\xi} = -\tfrac14(1+\eta)$$
$$N_{1,\eta} = -\tfrac14(1-\xi), \quad N_{2,\eta} = -\tfrac14(1+\xi)$$
$$N_{3,\eta} = \tfrac14(1+\xi), \quad N_{4,\eta} = \tfrac14(1-\xi)$$

- Determinant of Jacobian ($dA = \det[\,\mathbf{J}\,]d\xi d\eta$):

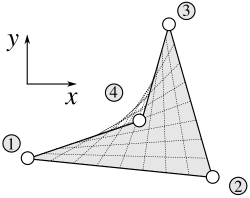
$$\det([\,\mathbf{J}\,]) =$$
$$\tfrac{1}{16}\left[((1-\eta)(x_2-x_1)+(1+\eta)(x_3-x_4))((1+\xi)(y_3-y_2)+(1-\xi)(y_4-y_1))-\right.$$
$$\left.-\;((1-\eta)(y_2-y_1)+(1+\eta)(y_3-y_4))((1+\xi)(x_3-x_2)+(1-\xi)(x_4-x_1))\right]$$

- **Warning:** to ensure $\det([\,\mathbf{J}\,]) > 0$ the element should remain convex

Parameteric space



Physical space



Solvers

Problem: Find $[\mathbf{u}]$ such that $\boxed{[\mathbf{K}]\,[\mathbf{u}] = [\mathbf{f}]}$, i.e. $[\mathbf{u}] = [\mathbf{K}]^{-1}\,[\mathbf{f}]$

- **Iterative solvers**
  The solution is approached iteratively, does not require much memory, restrictions to matrix type, sensitive to matrix conditioning, a preconditioner is often needed.
  - Gauss-Seidel method (GS)
  - Conjugate gradient method (CG)
  - Generalized minimum residual method (GMRES)
  - ...
- **Direct solvers**
  The solution is provided directly, no restrictions on matrix type, less sensitive to matrix conditioning, based on LU *or Cholesky decomposition*
  - Frontal
  - Sparse direct
  - ...

Example

- 3 bars in 2D
- 3 elements, 3 nodes, 6 dofs

References

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Thank you for your attention!