Computational Approach to Micromechanical Contacts

Lecture 3.
Mechanical Contact

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Outline

Boundary value problem in elasticity

- Reference and current configurations

$$\underline{x} = \underline{X} + \underline{u}$$

- Balance equation (strong form)

$$\nabla \cdot \underline{\underline{\sigma}} + \rho \underline{f}_v = 0, \forall \underline{x} \in \Omega^i$$

- Displacement compatibility

$$\underline{\underline{\varepsilon}} = \frac{1}{2}(\nabla \underline{u} + \underline{u}\nabla)$$

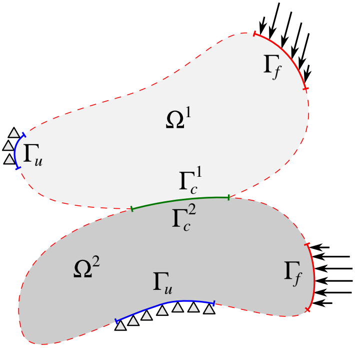
- Constitutive equation

$$\underline{\underline{\sigma}} = \frac{\partial W}{\partial \underline{\underline{\varepsilon}}}$$

- Boundary conditions

$$\text{Dirichlet: } \underline{u} = \underline{u}^0, \forall \underline{x} \in \Gamma_u$$

$$\text{Neumann: } \underline{n} \cdot \underline{\underline{\sigma}} = \underline{t}^0, \forall \underline{x} \in \Gamma_f$$



Two bodies in contact

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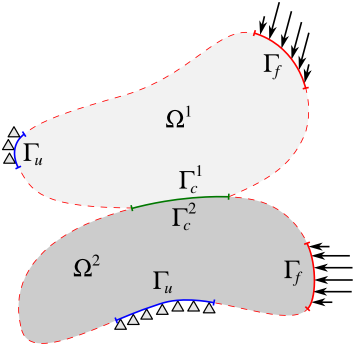


Two bodies in contact

- **Include contact conditions**

. . .

Intuitive conditions

1. No penetration

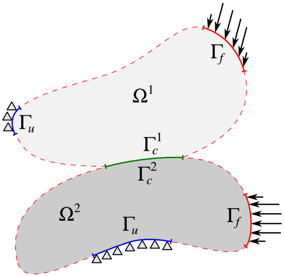
$$\Omega^1(t) \cap \Omega^2(t) = \emptyset$$

2. No adhesion

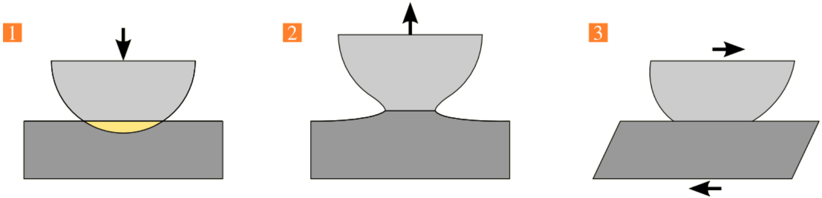
$$\underline{n} \cdot \underline{\underline{\sigma}} \cdot \underline{n} \le 0, \forall \underline{x} \in \Gamma_c^i$$

3. No shear stress

$$\underline{n} \cdot \underline{\underline{\sigma}} \cdot (\underline{\underline{I}} - \underline{n} \otimes \underline{n}) = 0, \forall \underline{x} \in \Gamma_c^i$$



Two bodies in contact



Intuitive contact conditions for frictionless and nonadhesive contact

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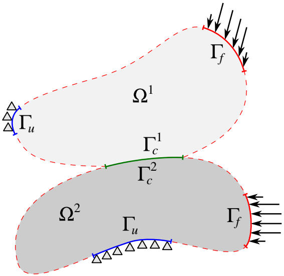
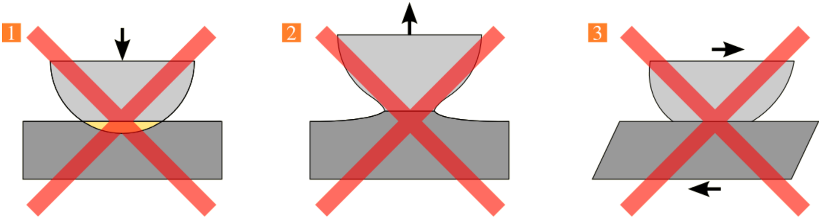


Two bodies in contact

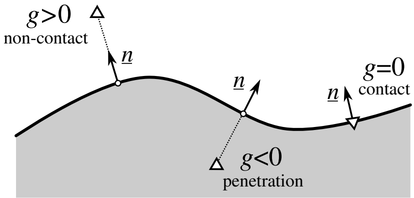


Intuitive contact conditions for frictionless and nonadhesive contact

Gap function

- **Gap function** g
  - gap = – penetration
  - asymmetric function
  - defined for
    - separation $g > 0$
    - contact $g = 0$
    - penetration $g < 0$
  - governs normal contact
- **Master and slave split**

  Gap function is determined for all slave points with respect to the master surface



Gap between a slave point and a master surface

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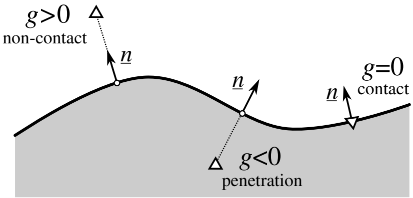
- **Master and slave split**

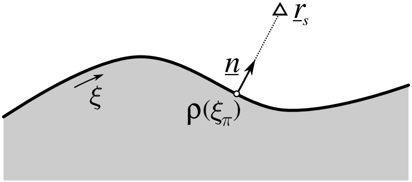
  Gap function is determined for all slave points with respect to the master surface
- **Normal gap**

$$g_n = \underline{n} \cdot \left[\underline{r}_s - \underline{\rho}(\xi_\pi) \right],$$

$\underline{n}$ is a unit normal vector, $\underline{r}_s$ slave point, $\underline{\rho}(\xi_\pi)$ projection point at master surface



Gap between a slave point and a master surface



Definition of the normal gap

Frictionless or normal contact conditions

- **No penetration**
  Always non-negative gap

  $$g \geq 0$$

- **No adhesion**
  Always non-positive contact pressure

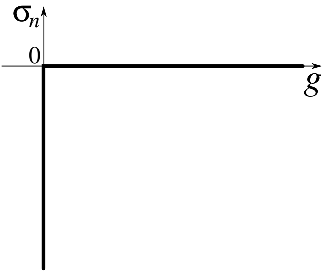
  $$\sigma_n^* \leq 0$$

- **Complementary condition**
  Either zero gap and non-zero pressure, or non-zero gap and zero pressure

  $$g\,\sigma_n = 0$$

- **No shear transfer** *(automatically)*

  $$\underline{\boldsymbol{\sigma}}_t^{**} = 0$$



Scheme explaining normal contact conditions

$\sigma_n^* = (\underline{\underline{\sigma}} \cdot \underline{n}) \cdot \underline{n} = \underline{\underline{\sigma}} : (\underline{n} \otimes \underline{n})$

$\underline{\boldsymbol{\sigma}}_t^{**} = \underline{\underline{\sigma}} \cdot \underline{n} - \sigma_n \underline{n} = \underline{n} \cdot \underline{\underline{\sigma}} \cdot \left(\underline{\underline{I}} - \underline{n} \otimes \underline{n}\right)$

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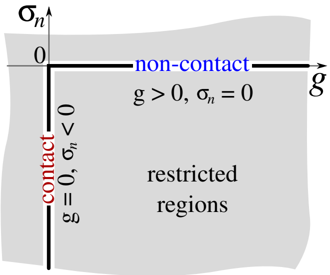

Improved scheme explaining normal contact conditions

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Frictionless or normal contact conditions

In **mechanics:**

Normal contact conditions

≡

Frictionless contact conditions

≡

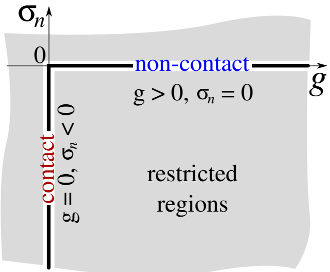
Hertz[1]*-Signorini*[2] *conditions*

≡

Hertz[1]*-Signorini*[2]*-Moreau*[3] *conditions*

also known in **optimization theory** as

Karush[4]*-Kuhn*[5]*-Tucker*[6] *conditions*



Improved scheme explaining normal contact conditions

$$g \ge 0, \qquad \sigma_n \le 0, \qquad g\sigma_n = 0$$

[1] **Heinrich Rudolf Hertz** (1857–1894) a German physicist who first formulated and solved the frictionless contact problem between elastic ellipsoidal bodies.

[2] **Antonio Signorini** (1888–1963) an Italian mathematical physicist who gave a general and rigorous mathematical formulation of contact constraints.

[3] **Jean Jacques Moreau** (1923) a French mathematician who formulated a non-convex optimization problem based on these conditions and introduced pseudo-potentials in contact mechanics.

[4] **William Karush** (1917–1997), [5] **Harold William Kuhn** (1925) American mathematicians,

[6] **Albert William Tucker** (1905–1995) a Canadian mathematician.

Contact problem

≈ **Problem**

Find such contact pressure

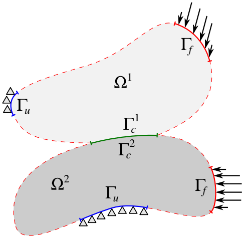
$$p = -\underline{n} \cdot \underline{\underline{\sigma}} \cdot \underline{n} \geq 0$$

which being applied at Γ_c^1 and Γ_c^2 results in

$$\underline{x}^1 = \underline{x}^2, \forall \underline{x}^1 \in \Gamma_c^1, \underline{x}^2 \in \Gamma_c^2$$

and evidently

$$\Omega^1(t) \cap \Omega^2(t) = \emptyset$$



Two bodies in contact

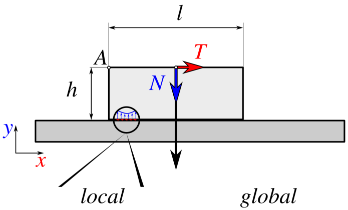
- Unfortunately, we do not know Γ_c^1 in advance, it is also an unknown of the problem.
- **Related problem**
  Suppose that we know p on Γ_c
  Then what is the corresponding displacement field $\underline{u}$ in Ω^i?

Evidence of friction

- Existence of frictional resistance is evident
- Independence of the nominal contact area

Think about adhesion and introduce a threshold in the interface τ_c

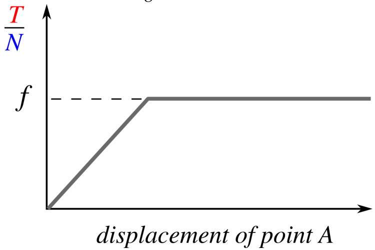
- Globally:
  - stick: $T < T_c(N)$
  - slip: $T = T_c(N)$
- From experiments:
  - Threshold $T_c \sim N$
  - Friction coefficient $f = |T_c/N|$
- Locally
  - stick: $\sigma_\tau < \tau_c(\sigma_n)$
  - slip: $\sigma_\tau = f\sigma_n$



σ_τ, σ_n | T, N

$f = \max(|\sigma_\tau/\sigma_n|)$ | $f = \max(|T/N|)$

Rectangle on a flat surface



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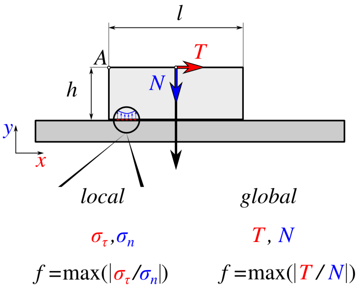
-  Torque



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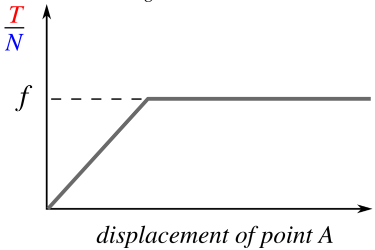
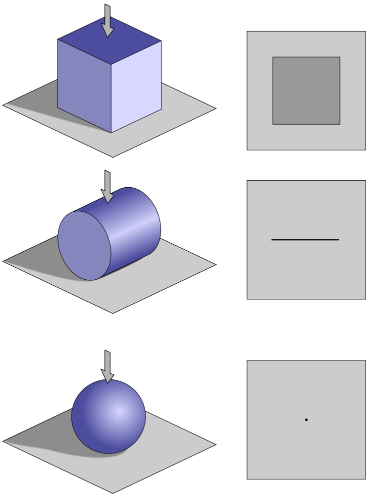


Types of contact

- Known contact zone
  - conformal geometry
    flat-to-flat, cylinder in a hole
  - initially non-conformal geometry but huge pressure resulting in full contact
- Unknown contact zone
  general case
- Point and line contact
- Frictionless
  conservative, energy minimization problem
- Frictional
  path-dependent solution, from the first touch to the current moment

Example



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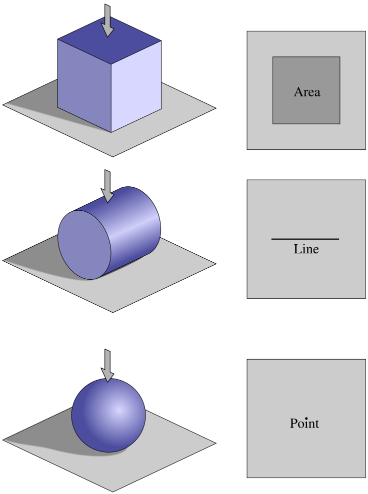

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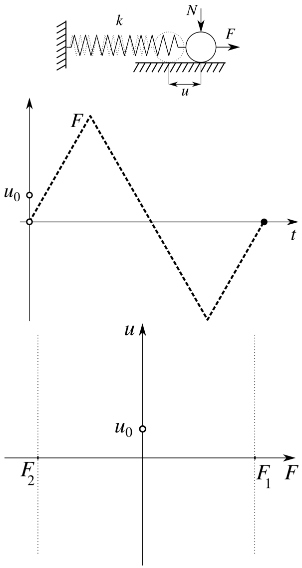
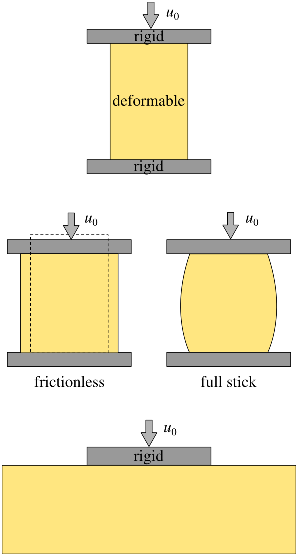


Flat geometry

- Compression of a cylinder
- Frictionless $u_z = u_0$
- Full stick conditions $\underline{u} = u_0\underline{e}_z$
- Rigid flat indenter $u_z = u_0$



Analogy with boundary conditions

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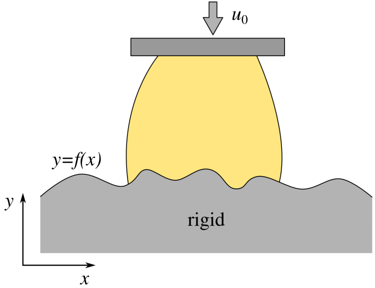
Curved geometry

- Polar/spherical coordinates
  $u_r = u_0$
- If frictionless contact on rigid surface $y = f(x)$ is retained by high pressure

  $(\underline{X} + \underline{u}) \cdot \underline{e}_y = f((\underline{X} + \underline{u}) \cdot \underline{e}_x)$



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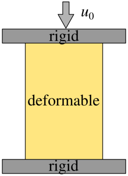
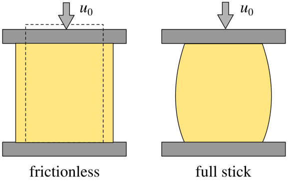
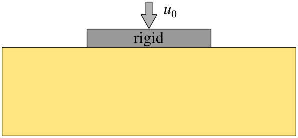
Transition to finite friction

- $\approx$ From full stick, decrease f by keeping $u_z = 0$ and by replacing in-plane Dirichlet BC by in-plane Neumann BC







Analogy with boundary conditions II

In general

- Type I: prescribed tractions 
  $p(x,y), \tau_x(x,y), \tau_y(x,y)$
- Type II: prescribed displacements
  $\underline{\boldsymbol{u}}(x,y)$
- Type III: tractions and displacements
  $u_z(x,y), \tau_x(x,y), \tau_y(x,y)$ or
  $p(x,y), u_x(x,y), u_y(x,y)$
- Type IV: displacements and relation between tractions
  $u_z(x,y), \tau_x(x,y) = \pm f p(x,y)$

Concentrated forces

- Normal force: in-plane stresses and displacements (plane strain)

$$\sigma_r = -\frac{2N}{\pi}\frac{\cos(\theta)}{r} \text{ or } \sigma_x = -\frac{2N}{\pi}\frac{x^2 y}{(x^2+y^2)^2},\ \sigma_y = -\frac{2N}{\pi}\frac{y^3}{(x^2+y^2)^2},\ \sigma_{xy} = -\frac{2N}{\pi}\frac{xy^2}{(x^2+y^2)^2}$$

$$u_r = \frac{1+\nu}{\pi E} N\cos(\theta)\left[2(1-\nu)\ln(r) - (1-2\nu)\theta\tan(\theta)\right] + C\cos(\theta)$$

$$u_\theta = \frac{1+\nu}{\pi E} N\sin(\theta)\left[2(1-\nu)\ln(r) - 2\nu + (1-2\nu)(1-2\theta\,\mathrm{ctan}(\theta))\right] - C\sin(\theta)$$

- Tangential force

$$\sigma_r = \frac{2T}{\pi}\frac{\sin(\theta)}{r} \text{ or } \sigma_x = -\frac{2T}{\pi}\frac{x^3}{(x^2+y^2)^2},\ \sigma_y = -\frac{2T}{\pi}\frac{xy^2}{(x^2+y^2)^2},\ \sigma_{xy} = -\frac{2T}{\pi}\frac{x^2 y}{(x^2+y^2)^2}$$

$$u_r = -\frac{1+\nu}{\pi E} T\sin(\theta)\left[2(1-\nu)\ln(r) - (1-2\nu)\theta\,\mathrm{ctan}(\theta)\right] - C\sin(\theta)$$

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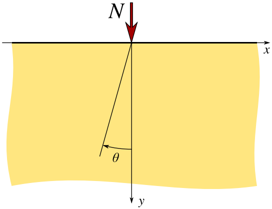
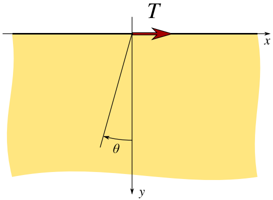
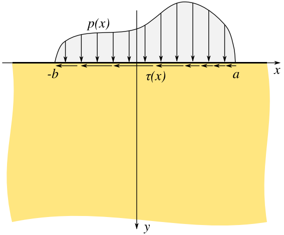




Distributed load

- Distributed tractions $p(x)dx = dN(x)$, $\tau(x)dx = dT(x)$
- Use superposition principle for the stress state and for displacements



Tractions on the surface

$$\sigma_x(x,y) = -\frac{2y}{\pi}\int_{-b}^{a}\frac{p(s)(x-s)^2\,ds}{((x-s)^2+y^2)^2} - \frac{2}{\pi}\int_{-b}^{a}\frac{\tau(s)(x-s)^3\,ds}{((x-s)^2+y^2)^2}$$

$$\sigma_y(x,y) = -\frac{2y^3}{\pi}\int_{-b}^{a}\frac{p(s)\,ds}{((x-s)^2+y^2)^2} - \frac{2y^2}{\pi}\int_{-b}^{a}\frac{\tau(s)(x-s)\,ds}{((x-s)^2+y^2)^2}$$

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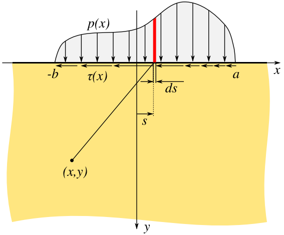

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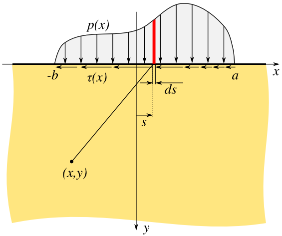
- Consider displacements on the surface

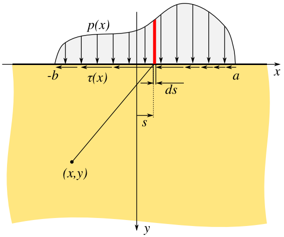


Tractions on the surface

$$u_x(x,0) = -\frac{(1-2\nu)(1+\nu)}{2E}\left[\int_{-b}^{x} p(s)\,ds - \int_{x}^{a} p(s)\,ds\right] - \frac{2(1-\nu^2)}{\pi E}\int_{-b}^{a} \tau(s)\ln|x-s|\,ds + C_1$$

Distributed load

- Distributed tractions $p(x)dx = dN(x)$, $\tau(x)dx = dT(x)$
- Use superposition principle for the stress state and for displacements
- Consider displacements on the surface
- Or rather their derivatives along the surface



Tractions on the surface

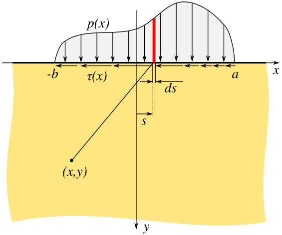
$$u_x(x,0) = -\frac{(1-2\nu)(1+\nu)}{2E}\left[\int_{-b}^{x} p(s)\,ds - \int_{x}^{a} p(s)\,ds\right] - \frac{2(1-\nu^2)}{\pi E}\int_{-b}^{a} \tau(s)\ln|x-s|\,ds + C_1$$

$$u_{x,x}(x,0) = -\frac{(1-2\nu)(1+\nu)}{E}p(x) - \frac{2(1-\nu^2)}{\pi E}\int_{-b}^{a}\frac{\tau(s)}{x-s}\,ds$$

Near-surface stress state

Distributed load

- Distributed tractions $p(x)dx = dN(x)$, $\tau(x)dx = dT(x)$
- Use superposition principle for the stress state and for displacements
- Consider displacements on the surface
- Or rather their derivatives along the surface

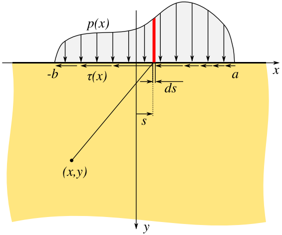


Tractions on the surface

$$u_y(x,0) = \frac{(1-2\nu)(1+\nu)}{2E}\left[\int_{-b}^{x}\tau(s)\,ds - \int_{x}^{a}\tau(s)\,ds\right] - \frac{2(1-\nu^2)}{\pi E}\int_{-b}^{a} p(s)\ln|x-s|\,ds + C_2$$

Distributed load

- Distributed tractions $p(x)dx = dN(x)$, $\tau(x)dx = dT(x)$
- Use superposition principle for the stress state and for displacements
- Consider displacements on the surface
- Or rather their derivatives along the surface



Tractions on the surface

$$u_y(x,0) = \frac{(1-2\nu)(1+\nu)}{2E}\left[\int_{-b}^{x} \tau(s)\,ds - \int_{x}^{a} \tau(s)\,ds\right] - \frac{2(1-\nu^2)}{\pi E}\int_{-b}^{a} p(s)\ln|x-s|\,ds + C_2$$

$$u_{y,x}(x,0) = \frac{(1-2\nu)(1+\nu)}{E}\tau(x) - \frac{2(1-\nu^2)}{\pi E}\int_{-b}^{a} \frac{p(s)}{x-s}\,ds$$

Rigid stamp problem

- Link displacement derivatives with tractions

$$\int_{-b}^{a} \frac{\tau(s)}{x-s}\,ds = -\frac{\pi(1-2\nu)}{2(1-\nu)}p(x) - \frac{\pi E}{2(1-\nu^2)}u_{x,x}(x,0)$$

$$\int_{-b}^{a} \frac{p(s)}{x-s}\,ds = \frac{\pi(1-2\nu)}{2(1-\nu)}\tau(x) - \frac{\pi E}{2(1-\nu^2)}u_{y,x}(x,0)$$

- If in contact interface we can prescribe $p, u_{x,x}$ or $\tau, u_{y,x}$, then the problem reduces to

$$\int_{-b}^{a} \frac{\mathcal{F}(x)}{x-s}ds = \mathcal{U}(x)$$

- The general solution (case $a = b$):

$$\mathcal{F}(x) = \frac{1}{\pi^2\sqrt{a^2-x^2}}\int_{-a}^{a}\frac{\sqrt{a^2-s^2}\,\mathcal{U}(s)\,ds}{x-s} + \frac{C}{\pi\sqrt{a^2-x^2}}, \quad C = \int_{-a}^{a}\mathcal{F}(s)ds$$

Rigid stamp problem

- Link displacement derivatives with tractions

$$\int_{-b}^{a} \frac{\tau(s)}{x-s}\,ds = -\frac{\pi(1-2\nu)}{2(1-\nu)}p(x) - \frac{\pi E}{2(1-\nu^2)}u_{x,x}(x,0)$$

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- If in contact interface we can prescribe $p, u_{x,x}$ or $\tau, u_{y,x}$, then the problem reduces to

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flat frictionless punch, consider P.V.

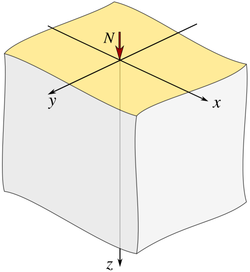
Three-dimensional problem

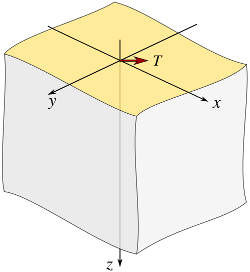
- Analogy to Flamant's problem
- Potential functions of Boussinesq
- Boussinesq problem
  concentrated normal force
- Cerruti problem
  concentrated tangential force
- Displacements decay as $\sim r^{-1}$

$$u_r(x,y,0) = -\frac{1-2\nu}{4\pi G}\frac{N}{\sqrt{x^2+y^2}}$$

$$u_z(x,y,0) = \frac{1-\nu}{4\pi G}\frac{N}{\sqrt{x^2+y^2}}$$

- Stress decay as $\sim r^{-2}$
- Superposition principle





Why is the sky dark at night?

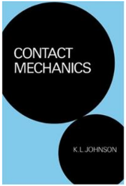
Why is the sky dark at night?

- Olbers' paradox or "dark night sky paradox"
- Two nominally-flat elastic half-spaces in contact
- At small scale they are rough with asperity density D
- Vertical displacement decay $u_z \sim 1/r$
- At every asperity, force F
- Sum up displacements induced by all forces

$$u_z \sim \int\limits_0^{2\pi} \int\limits_{r_0}^{R} \frac{F}{r} r \, dr \, d\phi \xrightarrow[R\to\infty]{} \infty$$

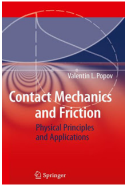
Classical contact problems

- Various problems with rigid flat stamps: *circular, elliptic, frictionless, full-stick, finite friction*
- Hertz theory
  normal frictionless contact of elastic solids 
  E_i, ν_i and $z_i = A_i x^2 + B_i y^2 + C_i xy, \quad i = 1,2$
- Wedges *(coin)* and cones
- Circular inclusion in a conforming hole
  Steuermann, 1939,Goodman, Keer, 1965
- Frictional indentation $z \sim x^n$
  Incremental approach Mossakovski, 1954
  self-similar solution Spence, 1968, 1975
- Adhesive contact Johnson et al, 1971, 1976
- Contact with layered materials (coatings)
- Elastic-plastic and viscoelastic materials
- Sliding/rolling of non-conforming bodies
  Cattaneo, 1938,Mindlin, 1949,Galin, 1953,Goryacheva, 1998
  Note: $u_r \sim (1-2\nu)/G$, so if $(1-2\nu_1)/G_1 = (1-2\nu_2)/G_2$ tangential tractions do not change normal ones



K.L. Johnson

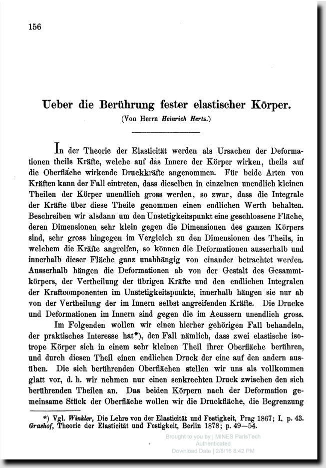
I.G. Goryacheva



V.L. Popov

Hertzian contact

- No friction, no adhesion
- Two elastic materials
  E_1, ν_1, E_2, ν_2
- Effective elastic modulus
  $$\frac{1}{E^*} = \frac{1-\nu_1^2}{E_1} + \frac{1-\nu_2^2}{E_2}$$
- Two parabolic surfaces
  $z_1 = ax_1^2 + by_1^2, z_2 = cx_2^2 + dy_2^2$
- Solids of revolution or cylinders R_1,R_2, effective curvature radius:
  $$\frac{1}{R^*} = \frac{1}{R_1} + \frac{1}{R_2}$$
- Displacement and contact radius (half-length):
  $\delta = a^2/R^*$
- Contact pressure:
  $$p(r) = p_0\sqrt{1-\frac{r^2}{a^2}},\; |r| < a$$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Ueber die Berührung fester elastischer Körper.

(Von Herrn *Heinrich Hertz.*)

In der Theorie der Elasticität werden als Ursachen der Deformationen theils Kräfte, welche auf das Innere der Körper wirken, theils auf die Oberfläche wirkende Druckkräfte angenommen. Für beide Arten von Kräften kann der Fall eintreten, dass dieselben in einzelnen unendlich kleinen Theilen der Körper unendlich gross werden, so zwar, dass die Integrale der Kräfte über diese Theile genommen einen endlichen Werth behalten. Beschreiben wir alsdann um den Unstetigkeitspunkt eine geschlossene Fläche, deren Dimensionen sehr klein gegen die Dimensionen des ganzen Körpers sind, sehr gross hingegen im Vergleich zu den Dimensionen des Theils, in welchem die Kräfte angreifen, so können die Deformationen ausserhalb und innerhalb dieser Fläche ganz unabhängig von einander betrachtet werden. Ausserhalb hängen die Deformationen ab von der Gestalt des Gesammtkörpers, der Vertheilung der übrigen Kräfte und den endlichen Integralen der Kraftcomponenten im Unstetigkeitspunkte, innerhalb hängen sie nur ab von der Vertheilung der im Innern selbst angreifenden Kräfte. Die Drucke und Deformationen im Innern sind gegen die im Aeussern unendlich gross.

Im Folgenden wollen wir einen hierher gehörigen Fall behandeln, der praktisches Interesse hat*), den Fall nämlich, dass zwei elastische isotrope Körper sich in einem sehr kleinen Theil ihrer Oberfläche berühren, und durch diesen Theil einen endlichen Druck der eine auf den andern ausüben. Die sich berührenden Oberflächen stellen wir uns als vollkommen glatt vor, d. h. wir nehmen nur einen senkrechten Druck zwischen den sich berührenden Theilen an. Das beiden Körpern nach der Deformation gemeinsame Stück der Oberfläche wollen wir die Druckfläche, die Begrenzung

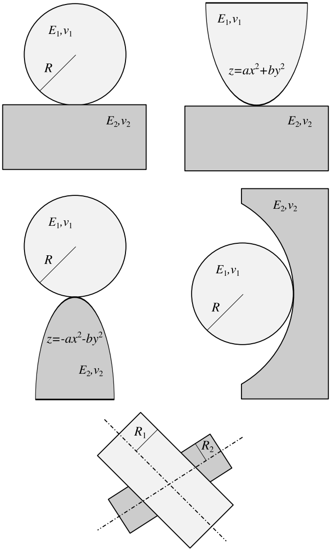
*) Vgl. *Winkler*, Die Lehre von der Elasticität und Festigkeit, Prag 1867; I, p. 43. *Grashof*, Theorie der Elasticität und Festigkeit, Berlin 1878; p. 49—54.

Original paper by Henrich Hertz "On the contact of elastic solids" (ENG trans.) (16 pages)

"His theory, worked out during the Christmas vacation 1880 at the age of 23(!), aroused considerable interest . . . " K.L. Johnson

Hertzian contact

- No friction, no adhesion
- Two elastic materials
  E_1, ν_1, E_2, ν_2
- Effective elastic modulus
  $$\frac{1}{E^*} = \frac{1-\nu_1^2}{E_1} + \frac{1-\nu_2^2}{E_2}$$
- Two parabolic surfaces
  $z_1 = ax_1^2 + by_1^2, z_2 = cx_2^2 + dy_2^2$
- Solids of revolution or cylinders R_1,R_2, effective curvature radius:
  $$\frac{1}{R^*} = \frac{1}{R_1} + \frac{1}{R_2}$$
- Displacement and contact radius (half-length):
  $\delta = a^2/R^*$
- Contact pressure:
  $$p(r) = p_0\sqrt{1-\frac{r^2}{a^2}},\ |r| < a$$



Geometries resolved in the framework of Hertz theory

Hertzian contact

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  E_1, ν_1, E_2, ν_2
- Effective elastic modulus
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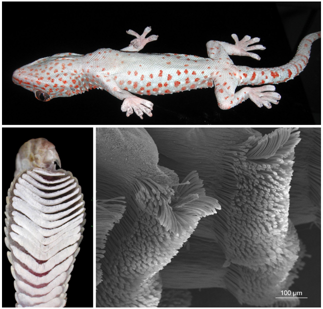
  $$\frac{1}{R^*} = \frac{1}{R_1} + \frac{1}{R_2}$$
- Displacement and contact radius (half-length):
  $\delta = a^2/R^*$
- Contact pressure:
  $$p(r) = p_0\sqrt{1-\frac{r^2}{a^2}},\ |r| < a$$
- Line contact (cylinders):
  $$a = \left(\frac{4PR^*}{\pi E^*}\right)^{1/2}$$
  $$p_0 = \frac{2P}{\pi a}$$
- Solids of revolution:
  $$a = \left(\frac{3PR^*}{4E^*}\right)^{1/3}$$
  $$p_0 = \frac{3P}{2\pi a^2}$$

- **Sources of adhesion in contact:** electrostatic forces (e.g. formation of a double electric layer), capilary adhesion, van der Waals forces, atmospheric pressure, magnetism, or in a long term, formation of stronger bounds.

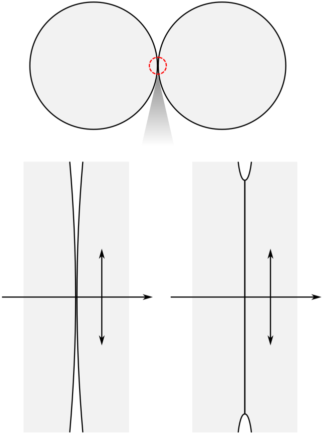


Adhesive contact

- Theory of rubber adhesion JKR[1]
- Approximate analysis
- Surface energy $U_s = -\pi a^2 \gamma$ with γ surface energy per unit area.
- Associated adhesive force $F_s = -\dfrac{\partial U_s}{\partial x}$
- Where x is approximately Hertzian approaching $x \approx a^2/R^*$
- Then the adhesive force:

$$\boxed{F_s \approx \pi \gamma R^*}$$

- Adhesive force is independent of contact area and elastic properties of materials.

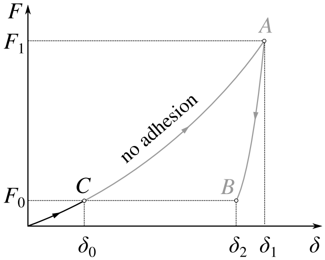


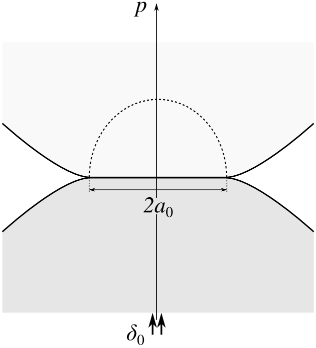
[1] Johnson, Kendall, Roberts, PRSL A, 324 (1971).

Adhesive contact: exact JKR[1] analysis I

- Bring in non-adhesive contact (O→A):
  - contact radius: $a_1^3 = \frac{3R^*}{4E^*}F_1$
  - contact pressure: $p(r) = \frac{3F_1}{2\pi a_1^2}\sqrt{1-\frac{r^2}{a_1^2}}$
  - approaching:

$$\delta_1 = a_1^2/R^* = \left[\frac{9}{16E^{*2}R^*}\right]^{1/3} F_1^{2/3}$$





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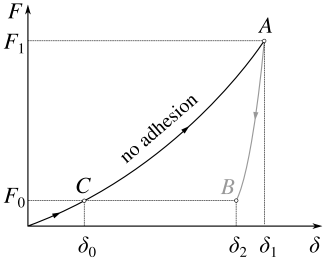

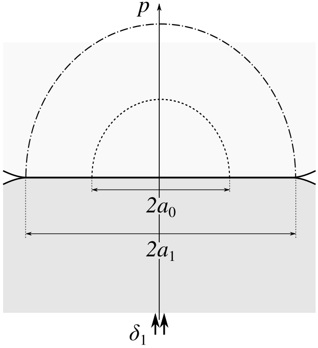

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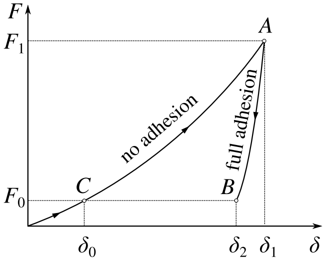
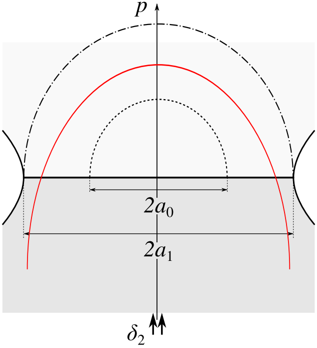
  - contact pressure: $p(r) = \frac{3F_1}{2\pi a_1^2}\sqrt{1 - \frac{r^2}{a_1^2}}$
  - approaching:

$$\delta_1 = a_1^2/R^* = \left[\frac{9}{16E^{*2}R^*}\right]^{1/3} F_1^{2/3}$$

- Fix the contact area and reduce the load[2] down to F_0 (A→B):
- New pressure distribution arises: $p'(r)$
- To keep points in contact, the induced extra displacement should be constant along the contact area $u_z = \mathrm{const}$
- Thus the corresponding pressure is $p'(r) \sim 1/\sqrt{1 - r^2/a_1^2}$





[1] Johnson, Kendall, Roberts, PRSL A, 324 (1971).

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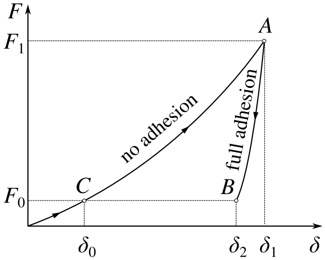
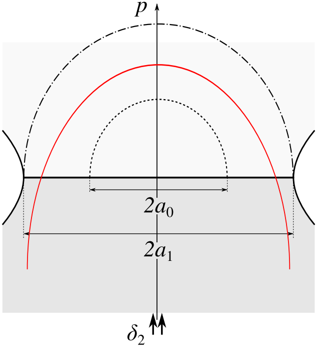
Adhesive contact: exact JKR[1] analysis II

- Energy of the system U = stored elastic energy U_e + energy of the applied load U_l + surface energy U_s
- Stored energy on O→A:

$$U_e(A) = \int\limits_0^{\delta_1} F\, d\delta = \frac{2}{5}\left(\frac{9}{16E^{*2}R^*}\right)^{1/3} F_1^{5/3}$$

[1] Johnson, Kendall, Roberts, PRSL A, 324 (1971).





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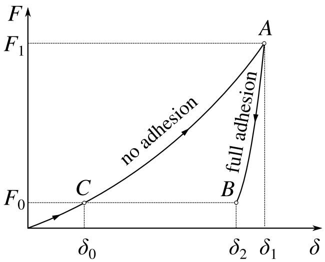
- Energy of the system U = stored elastic energy U_e + energy of the applied load U_l + surface energy U_s
- Stored energy on O→A:

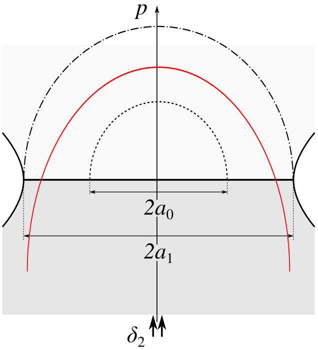
$$U_e(A) = \int_0^{\delta_1} F\,d\delta = \frac{2}{5}\left(\frac{9}{16E^{*2}R^*}\right)^{1/3} F_1^{5/3}$$

- Lost energy on A→B under condition of fixed contact area:

$$U_e(AB) = \int_{\delta_1}^{\delta_2} F\,d\delta = \frac{1}{4}\frac{F_0^2 - F_1^2}{E^* a_1}$$

Note[2] at this path: $\delta = \dfrac{1}{2}\dfrac{F}{E^* a_1}$





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Adhesive contact: exact JKR[1] analysis II

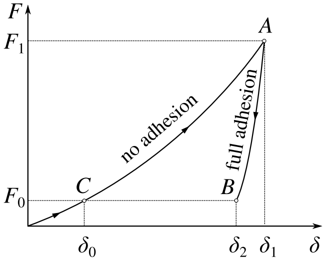
- Energy of the system U = stored elastic energy U_e + energy of the applied load U_l + surface energy U_s
- Stored energy on O→A:

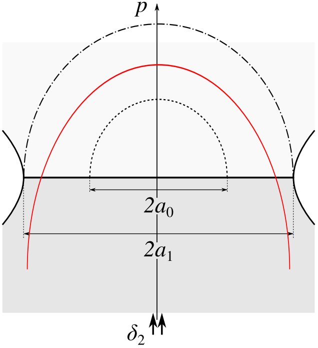
$$U_e(A) = \int\limits_0^{\delta_1} F\,d\delta = \frac{2}{5}\left(\frac{9}{16E^{*2}R^*}\right)^{1/3} F_1^{5/3}$$

- Lost energy on A→B under condition of fixed contact area:

$$U_e(AB) = \int\limits_{\delta_1}^{\delta_2} F\,d\delta = \left(\frac{1}{3\cdot 16E^{*2}R^*}\right)^{1/3} \frac{(F_0^2 - F_1^2)}{F_1^{1/3}}$$

Note[2] at this path: $\delta = \dfrac{1}{2}\dfrac{F}{E^* a_1}$





[1] Johnson, Kendall, Roberts, PRSL A, 324 (1971).

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Adhesive contact: exact JKR[1] analysis II

- Energy of the system U = stored elastic energy U_e + energy of the applied load U_l + surface energy U_s
- Stored energy on O→A:

$$U_e(A) = \int\limits_0^{\delta_1} F\,d\delta = \frac{2}{5}\left(\frac{9}{16E^{*2}R^*}\right)^{1/3} F_1^{5/3}$$

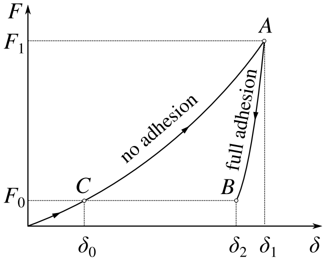
- Lost energy on A→B under condition of fixed contact area:

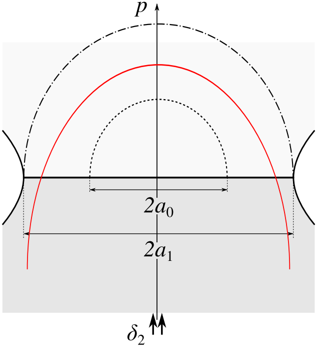
$$U_e(AB) = \int\limits_{\delta_1}^{\delta_2} F\,d\delta = \left(\frac{1}{3\cdot 16E^{*2}R^*}\right)^{1/3} \frac{(F_0^2 - F_1^2)}{F_1^{1/3}}$$

Note[2] at this path: $\delta = \dfrac{1}{2}\dfrac{F}{E^* a_1}$

- Remained stored energy:

$$U_e(OAB) = \frac{F_1^{5/3} + 5F_0^2 F_1^{-1/3}}{5(48E^{*2}R)^{1/3}}$$





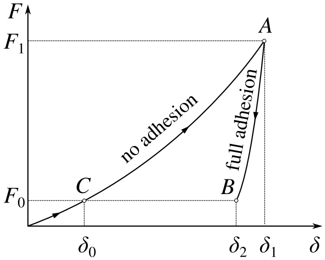
[1] Johnson, Kendall, Roberts, PRSL A, 324 (1971).

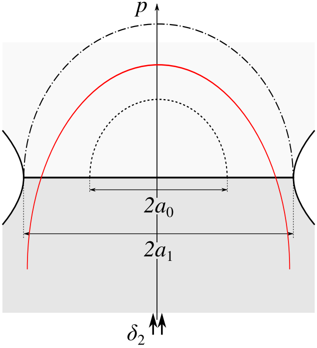
[2] Johnson, British J Appl Phys, 9 (1958).

Adhesive contact: exact JKR[1] analysis III

- Energy of the system U = stored elastic energy U_e + energy of the applied load U_l + surface energy U_s
- Mechanical potential energy of the applied load:

$$U_l = -F_0\delta_2 = -F_0\left(\delta_1 - \frac{1}{2}\frac{F_1 - F_0}{E^* a_1}\right)$$





[1] Johnson, Kendall, Roberts, PRSL A, 324 (1971).

- Energy of the system U = stored elastic energy U_e + energy of the applied load U_l + surface energy U_s
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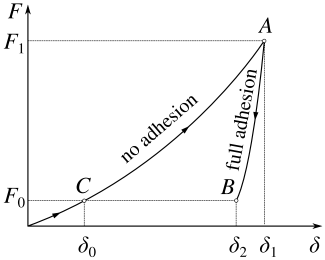
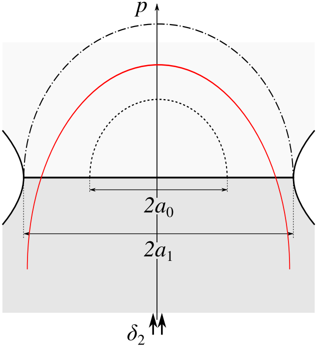
$$U_l = -F_0\left(\delta_1 - \frac{1}{2}\frac{F_1 - F_0}{E^*}\left[\frac{4E^*}{3R^*F_1}\right]^{1/3}\right)$$



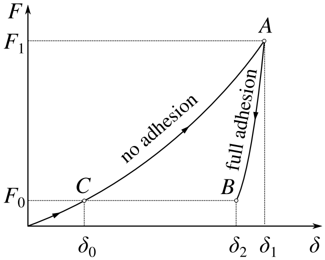


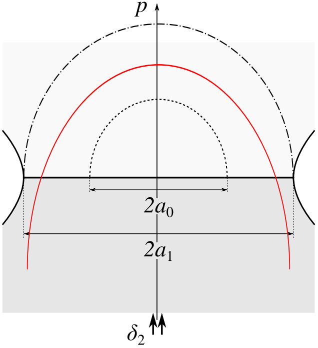
[1] Johnson, Kendall, Roberts, PRSL A, 324 (1971).

Adhesive contact: exact JKR[1] analysis III

- Energy of the system U = stored elastic energy U_e + energy of the applied load U_l + surface energy U_s
- Mechanical potential energy of the applied load:

$$U_l = -F_0\left(\left[\frac{9}{16E^{*2}R^*}\right]^{1/3} F_1^{2/3} - \frac{1}{2}\frac{F_1 - F_0}{E^*}\left[\frac{4E^*}{3R^*F_1}\right]^{1/3}\right)$$





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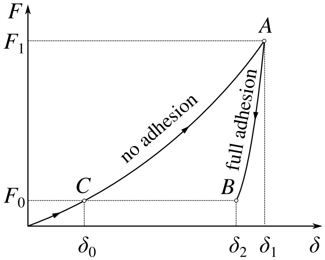
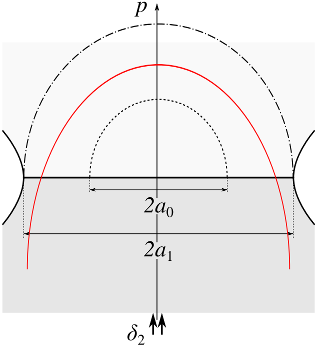
- Finally

$$U_l = -F_0\left[\frac{1}{48E^{*2}R^*}\right]^{1/3}\left(F_1^{2/3} + 2F_0F_1^{-1/3}\right)$$





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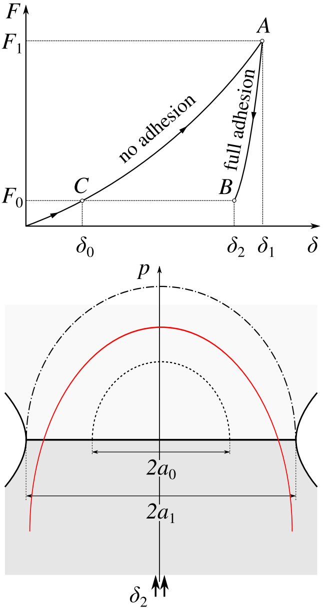
- Surface energy:

$$U_s = -\gamma\pi a_1^2 = -\gamma\pi\left[\frac{3R^*F_1}{4E^*}\right]^{2/3}$$



[1] Johnson, Kendall, Roberts, PRSL A, 324 (1971).

Adhesive contact: exact JKR[1] analysis IV

- Energy minimization

$$\frac{dU}{da_1} = \frac{d}{da_1}\left[U_e + U_l + U_s\right] = 0$$

- Equivalently

$$dU/da_1 = 0 \quad \Leftrightarrow \quad dU/dF_1 = 0$$

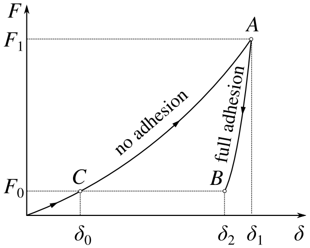
$$F_1^2 - 2F_1(F_0 + 3\gamma\pi R^*) + F_0^2 = 0$$

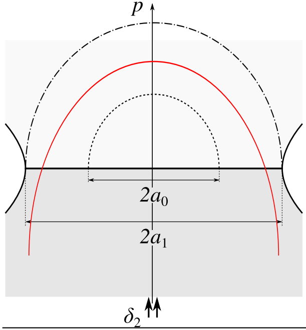
- The root ensuring stable minimum:

$$F_1 = F_0 + b + \sqrt{2bF_0 + b^2}, \quad b = 3\gamma\pi R^*$$

- Adhesive contact area:

$$a^3(F_0) = \frac{3R^*}{4E^*}F_1(F_0)$$





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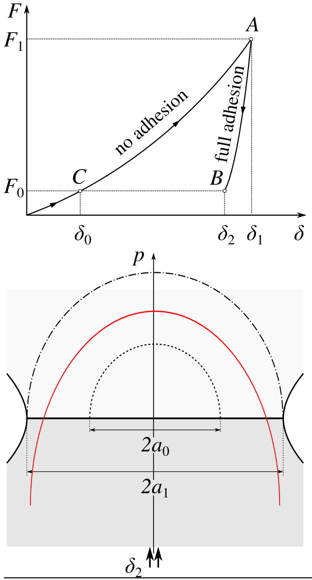

[1] Johnson, Kendall, Roberts, PRSL A, 324 (1971).

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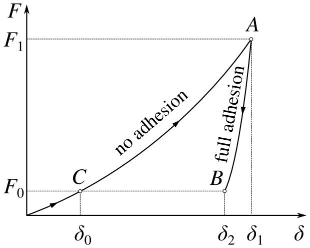
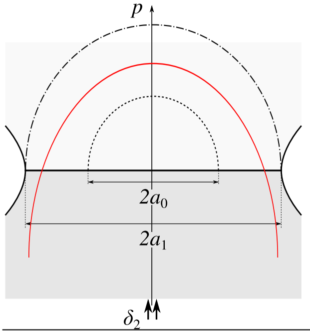
$$\frac{a^3(F_0)}{R^{*3}} = \frac{3}{4E^*R^{*2}}\left[F_0 + b + \sqrt{2bF_0 + b^2}\right]$$





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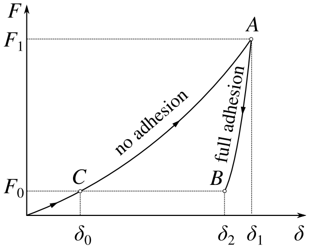
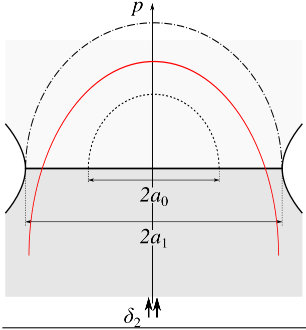
- Adhesive contact area:

$$a' = \left[F' + b' + \sqrt{2b'F' + b'^2}\,\right]^{1/3}$$

$$a' = \frac{a}{R^*}, \quad F' = \frac{3F}{4E^*R^{*2}}, \quad b' = \frac{9\gamma\pi}{4E^*R}$$





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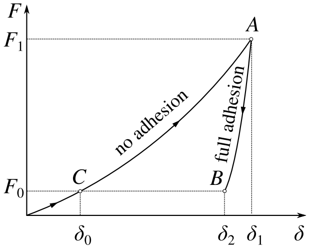
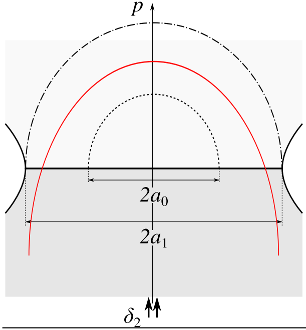
$$a' = \frac{a}{R^*}, \quad F' = \frac{3F}{4E^*R^{*2}}, \quad b' = \frac{9\gamma\pi}{4E^*R}$$

- Maximal tensile force (adhesive):

$$\min\{F\} = -b/2 = -\frac{3}{2}\pi\gamma R$$





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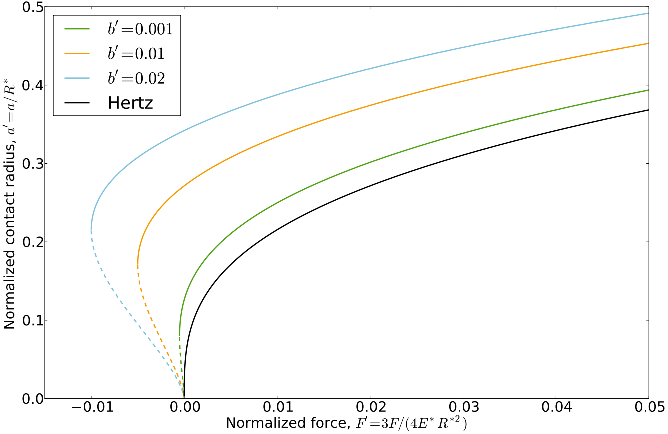
Normalized contact radius: $a' = \left[F' + b' + \sqrt{2b'F' + b'^2}\,\right]^{1/3}$

with $a' = \dfrac{a}{R^*}$, $F' = \dfrac{3F}{4E^*R^{*2}}$, $b' = \dfrac{9\gamma\pi}{4E^*R}$



- No friction, no adhesion
- Two elastic materials
  $E_1, \nu_1, E_2, \nu_2 \rightarrow E^*$
- Interface profile
  $y = \Delta(1 - \cos(2\pi x/\lambda)$
- Full contact pressure: $p^* = \pi E^* \Delta/\lambda$
- Contact length:

  $2a/\lambda = (2/\pi)\sqrt{\arcsin(\bar{p}/p^*)}$, where $\bar{p}$ is the applied pressure.
- Contact pressure distribution:

$$p(x,a) = \frac{2\bar{p}\cos(\pi x/\lambda)}{\sin^2(\pi a/\lambda)\sqrt{\sin^2(\pi a/\lambda - \sin^2(\pi x/\lambda)}}$$

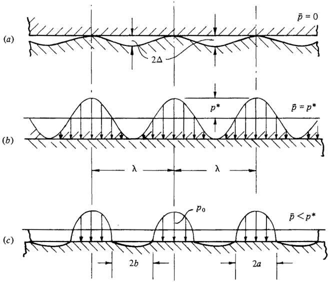


Illustration from K.L. Johnson (1985)

[1] Westergaard, ASME J Appl Mech (1939)

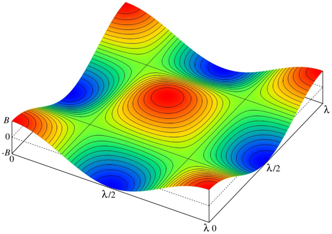
Westergaard solution: 3D extension

- Surface

  $z = \Delta(2 - \cos(2\pi x/\lambda) - \cos(2\pi y/\lambda))$
- Full contact pressure: $p^* = 2\pi E^* \Delta/\lambda$
- Contact area for $a \ll \lambda$:

  $$\frac{\pi a^2}{\lambda^2} = \pi \left(\frac{3\bar{p}}{8\pi p^*}\right)^{2/3}$$
- Non-contact area for $b \ll \lambda$:

  $$\frac{\pi b^2}{\lambda^2} = \frac{3}{2\pi}\left(1 - \frac{\bar{p}}{p^*}\right)$$



Bi-wavy surface[2]

[1] Johnson, Greenwood, Higginson, Int J Mech Sci (1985)

[2] Yastrebov, Anciaux, Molinari, Tribol Lett 56 (2014)

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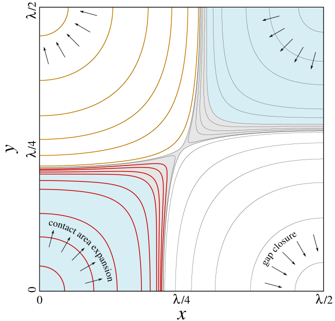

Contact area evolution[2]

[1] Johnson, Greenwood, Higginson, Int J Mech Sci (1985)

[2] Yastrebov, Anciaux, Molinari, Tribol Lett 56 (2014)

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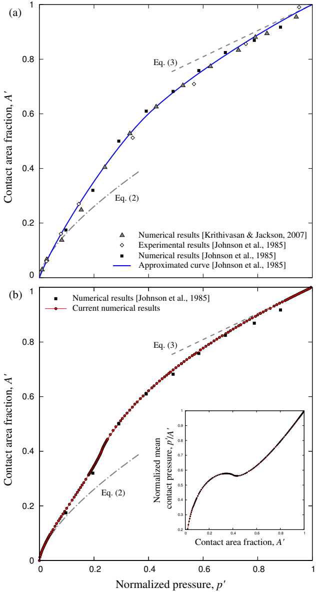

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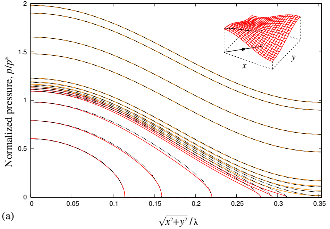

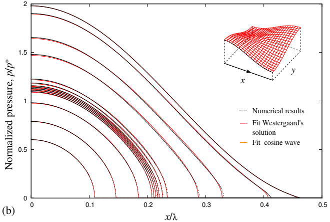

Pressure distribution along $y = x$ and $y = 0$[2]

[1] Johnson, Greenwood, Higginson, Int J Mech Sci (1985)

[2] Yastrebov, Anciaux, Molinari, Tribol Lett 56 (2014)

Thank you for your attention!