One degree-of-freedom frictional system

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1 Set-up

Consider a one degree-of-freedom frictional system depicted in Fig. 1. Spring's stiffness is k, externally applied force $F(t)$ changes in time, normal force N remains constant, u denotes the location of the contact point along OU axis. The contact interface is governed by Coulomb's law with the friction coefficient f. The spring is stretched by displacement u_0 from its equilibrium state $u = 0$ and is brought in frictional contact by normal force N. Note that after removing the displacement control the point remains in stick conditions. The external forcing changes F with time. Inertial effects are neglected.

2 Task

How does the position of the contact point u changes if the external load is applied according to the following law (see Fig. 2):

$$F(t) = \begin{cases} F_c t/t_1, & 0 \le t < t_1 \\ F_c(2t_1 - t)/t_1, & t_1 \le t < 3t_1 \\ F_c(t - 4t_1)/t_1, & 3t_1 \le t < 4t_1 \end{cases}$$

In two words it corresponds to the load F monotonically increasing from zero to F_c, then monotonically decreasing to $-F_c$, and finally monotonically increasing again to 0. Since the spring is linearly elastic (rate independent) as well as the friction law, the rate at which the force changes do not alter the solution.

3 Solution

- The frictional force F_t is bounded by a product of the friction coefficient f and the absolute value of the normal force $|N|$:

$$|F_f| \le f|N|$$

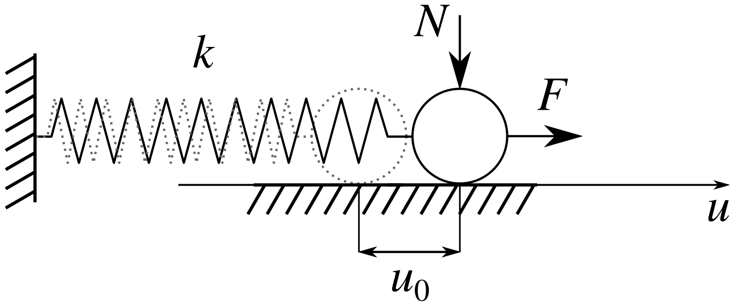

Figure 1: One degree-of-freedom frictional system

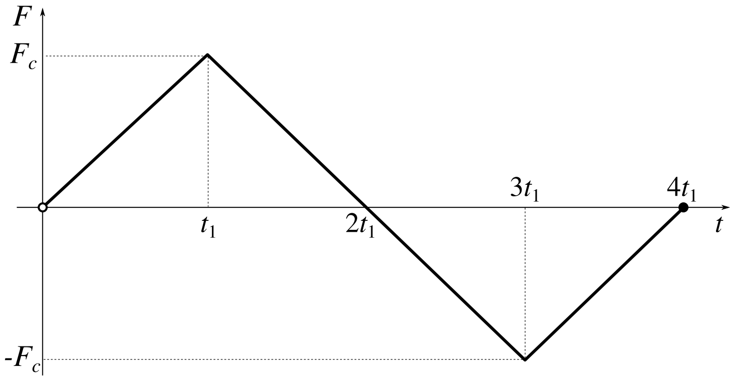

Figure 2: External force acting on the one degree-of-freedom frictional system

In the *stick state*, the frictional force can point either in left or right direction in order to equilibrate other forces acting in the system to ensure that the contact point does not move $\dot{u} = 0$. In the *slip state*, the frictional force is opposite to the direction of relative motion[1]

$$\operatorname{sign}(F_f) = -\operatorname{sign}(\dot{u}),$$

where

$$\operatorname{sign}(x) = \begin{cases} 1, & x > 0; \\ 0, & x = 0; \\ -1, & x < 0. \end{cases}$$

The absolute value of the frictional force is equivalent to $f|N|$.

- Since according to the problem set-up, in the beginning (at $t = 0$) the mass sticks to the ground we can deduce that the reaction force in the spring $F_s = -ku_0$ (produced by the initial displacement u_0) could be equilibrated by the frictional force $F_s + F_f = 0$. Therefore, since the frictional force is bounded by $|F_f| \le f|N|$, then the initial displacement is also bounded:

$$k|u_0| \le f|N| \quad \Leftrightarrow \quad -f|N|/k \le u_0 \le f|N|/k.$$

[1] Since we assume that the rigid flat ground does not move nor deform, the relative motion reduces to the absolute motion of the mass point.

- For increasing external force $F \sim t$, before the frictional force reaches its limit, the system will remain in stick condition $u = u_0$, the equilibrium of forces can be written as follows:

$$-ku_0 + F_f + F = 0 \quad \Leftrightarrow \quad -ku_0 + F_f + F_c t/t_1 = 0,$$

the frictional force adjusts itself to ensure the equilibrium, i.e.

$$F_f = ku_0 - F_c t/t_1.$$

- When frictional force reaches its limit[2], i.e.

$$F_f = -f|N| = ku_0 - F_c t/t_1,$$

which happens at time

$$t_r = (ku_0 + f|N|)t_1/F_c,$$

then the contact point will start to move in the right direction following the increasing force F. The corresponding external force at $t = t_r$ is given by

$$F_r = ku_0 + f|N|$$

If however, $t_r < t_1$ no slip will start up to the moment when external force reaches its maximum value. Later, when the external force inverses its sign ($t > 2t_1$) the stick state will be preserved up to the minimal value $F = -F_c$ if $|-ku_0 - F_c| < f|N|$, otherwise the point will start to slip to the left.

- Let us assume that $t_r < t_1$ and the point starts to slip before the external force reaches its maximum value. Then the increasing external force can no longer be balanced by frictional force (which remains constant in slip), but will be balanced by the reaction force in the spring that will stretch more. Since the spring is linear, it will stretch proportionally to the difference between the external force and the frictional force:

$$k\Delta u = F_c(t - t_r)/t_1$$

The force equilibrium can be written as:

$$-k[u_0 + \Delta u] - f|N| + F_c t/t_1 = 0$$

- At reaching $t = t_1$ (corresponding to the maximal external force), the force equilibrium takes the following form:

$$-k[u_0 + \Delta u^*] - f|N| + F_c = 0 \qquad (*)$$

from which the total slip can be found

$$\Delta u^* = (F_c - f|N|)/k - u_0.$$

[2]We take the frictional force with the negative sign since when the point starts to slip to the right, the frictional force should point to the left.

- When the external force inverses at $t = t_1$ and start to decrease to zero in the interval $t_1 \le t \le 2t_1$, three scenarios can be considered:

  1. **Scenario 1**: Mass-point continues to move to the right.
  2. **Scenario 2**: Mass-point moves to the left.
  3. **Scenario 3**: Mass-point sticks to the current location and does not move.

  To choose the *only possible* scenario, let us denote by δu the point displacement increment from its location found at $t = t_1$, by δt we denote the time increment $\delta t = t - t_1$, by $-\delta F < 0$ we denote the external force increment which is negative and is given by $-F_c \delta t / t_1$; finally, δF_f will denote the increment of the frictional force. We can rewrite (*) at $t = t_1 + \delta t$ as follows:

  $$-k\left[u_0 + \Delta u^* + \delta u\right] - f|N| + \delta F_f + F_c - \delta F = 0$$

  by subtracting (*) from this equation we obtain:

  $$-k\delta u + \delta F_f - \delta F = 0. \qquad (^{**}).$$

  Scenario 1 assumes that $\delta u > 0$, then from (**) it follows that the frictional force increment takes the form:

  $$\delta F_f = \delta F + k\delta u > 0.$$

  Then the total frictional force is given by $F_f = -f|N| + \delta F_f$. It means that its absolute value becomes less than the frictional limit $|F_f| < f|N|$. However, in this case the point should switch to the stick state, since in slip, the frictional force should be at its extreme value. However, since we assumed in this scenario that the point continues its motion to the right, we obtain a contradiction. So this scenario is impossible.

  Scenario 2 assumes that $\delta u < 0$, then (**) is given by

  $$-k\delta u + \delta F_f - \delta F = 0.$$

  At the same time, to let the point move to the right the frictional force should take the value $f|N|$ and should be directed to the right (opposite to the point motion), i.e. $\delta F_f = 2|f|N$. But it is unphysical that frictional force switches abruptly. Consider, for example, a box that we move forward with a force F, it slides smoothly at constant velocity v. Now, imagine that we slightly decreased the force $F - \delta F$, and the box slipped immediately back, because the frictional force which opposed the motion switched the sign. It does not happen. Such an abrupt switch would also imply that the following equation is satisfied:

  $$-k\delta u + 2|f|N - \delta F = 0,$$

  which has no solution for small time and displacement increments, i.e. for both $\delta t \to 0$ and $\delta u \to 0$. To satisfy this equation the point mass should jump to the left by

  $$\delta u = (2|f|N - \delta F)/k.$$

Hence, if the frictional force is allowed to switch abruptly, the equilibrium cannot be ensured for arbitrary external force history. All this is unphysical and thus Scenario 2 must be excluded.

Scenario 3 assumes that $\delta u = 0$, then (**) transforms into

$$\delta F_f - \delta F = 0, \qquad (***)$$

which implies that frictional force increment balances the external force increment. The total frictional force is $F_f = -f|N| + \delta F$, and its absolute value is below the frictional limit $|F_f| < f|N|$, thus the point should remain in stick, and it is what was assumed in this scenario. So no contradiction is found here, no abrupt changes in forces happen, and equilibrium can be satisfied at each time moment. So Scenario 3 is the only possible.

- At time t_l such that $t_1 < t_l \le 3t_1$, the frictional force increment will reach its maximum value $\delta F_f = 2f|N|$, i.e. the total frictional force will point to the right and be equal by value $f|N|$. To find t_l, we replace the external force increment δF by $F_c \delta t / t_1$ and obtain the following equation:

  $$2f|N| = F_c \delta t / t_1,$$

  which gives

  $$\delta t = 2f|N| t_1 / F_c,$$

  so the total time of the start of backward motion is given by

  $$t_l = t_1 + \delta t = t_1 \left[1 + \frac{2f|N|}{F_c} \right]. \qquad (\star)$$

  The resulting force is given by:

  $$F_l = F_c - 2f|N|.$$

  If the initial position $u_0 > 0$, then it can be shown that $f|N| < F_c$ and thus the inversion of motion will always happen before $t = 3t_1$, since the square brackets in $(\star)$ is bounded as

  $$1 \le \left[1 + \frac{2f|N|}{F_c} \right] \le 3.$$

  For $t_l \le t \le 3t_1$ the point will slip to the left and squeeze the spring which will balance the excess of the external force.

- At time $t > t_3$ the point will stick again and retain its position until $t = 4t_1$.

Graphically, the solution is depicted in Fig. 3. The easiest way to solve this problem geometrically is to plot the solution in force-displacement coordinates as in Fig. 3(a). Two parallel lines of slope $1/k$, symmetric over $F = 0$ and spaced by $2f|N|$ mark lines on which the slip will happen. In between them, the stick zone is located. The initial stick zone is coloured in light green: any pair of force and displacement will result in stick state. On the left line, the motion can occur only to the left, on the right line, it can occur only in the right direction, in between no motion is possible. The resulting solution in displacement-time coordinates is given in Fig. 3(b).

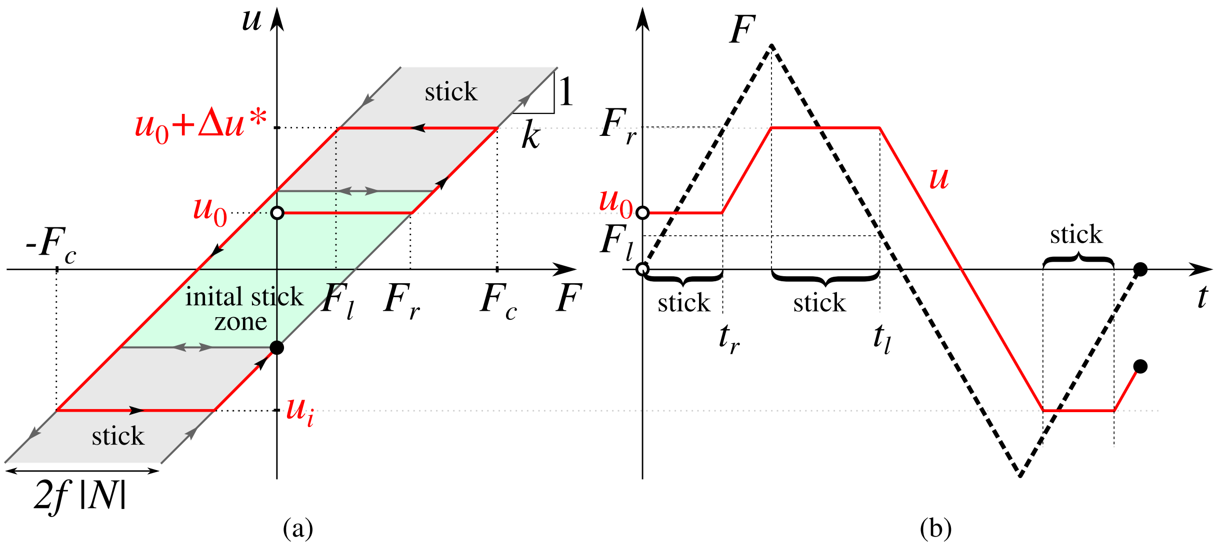

Figure 3: Solution of the problem: (a) solution in force-displacement space, initial stick zone is coloured in light green; (b) force and displacement are depicted as functions of time.