

Contact Mechanics and Elements of Tribology

Lecture 2. *Mechanical Contact*

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Outline

Lecture 2

- 1 Balance equations
- 2 Intuitive notions
- 3 Formalization of frictionless contact
- 4 Evidence friction
- 5 Contact types
- 6 Analogy with boundary conditions

Lecture 3

- 1 Flamant, Boussinesq, Cerruti
- 2 Displacements and tractions
- 3 Classical elastic problems

Boundary value problem in elasticity

- Reference and current configurations

$$\underline{x} = \underline{X} + \underline{u}$$

- Balance equation (strong form)

$$\nabla \cdot \underline{\underline{\sigma}} + \rho \underline{f}_v = 0, \forall \underline{x} \in \Omega^i$$

- Displacement compatibility

$$\underline{\underline{\epsilon}} = \frac{1}{2} (\nabla \underline{u} + \underline{u} \nabla)$$

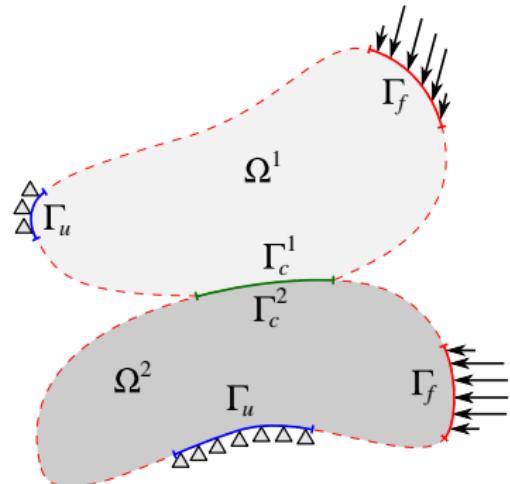
- Constitutive equation

$$\underline{\underline{\sigma}} = W'(\underline{\underline{\epsilon}})$$

- Boundary conditions

$$\text{Dirichlet: } \underline{u} = \underline{u}^0, \forall \underline{x} \in \Gamma_u$$

$$\text{Neumann: } \underline{n} \cdot \underline{\underline{\sigma}} = \underline{t}^0, \forall \underline{x} \in \Gamma_f$$



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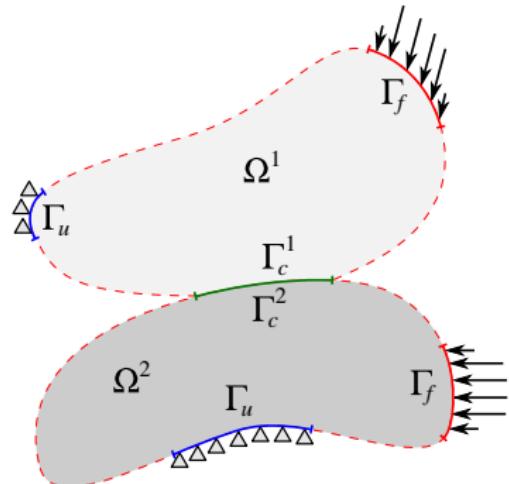
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Two bodies in contact

- **Include contact conditions**

...

Intuitive conditions

1 No penetration

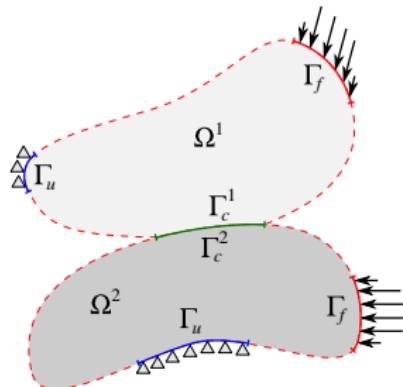
$$\Omega^1(t) \cap \Omega^2(t) = \emptyset$$

2 No adhesion

$$\underline{n} \cdot \underline{\sigma} \cdot \underline{n} \leq 0, \forall \underline{x} \in \Gamma_c^i$$

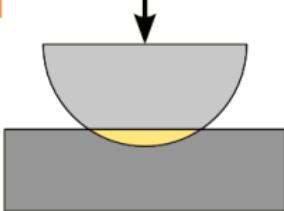
3 No shear stress

$$\underline{n} \cdot \underline{\sigma} \cdot (\underline{I} - \underline{n} \otimes \underline{n}) = 0, \forall \underline{x} \in \Gamma_c^i$$

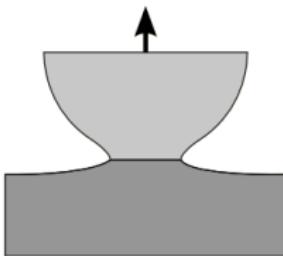


Two bodies in contact

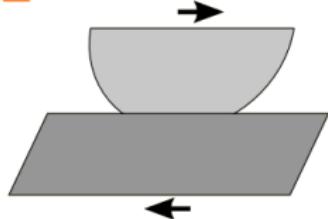
1



2



3



Intuitive contact conditions for frictionless and nonadhesive contact

Intuitive conditions

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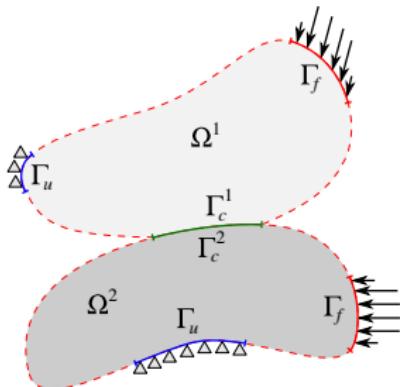
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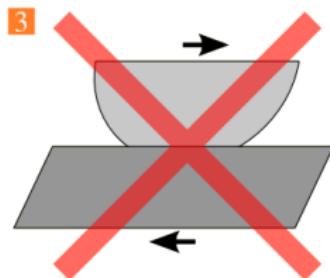
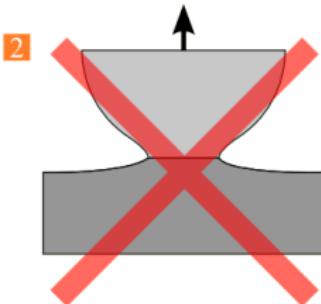
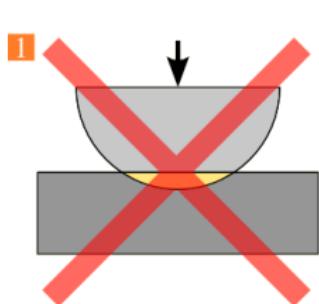
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Two bodies in contact



Intuitive contact conditions for frictionless and nonadhesive contact

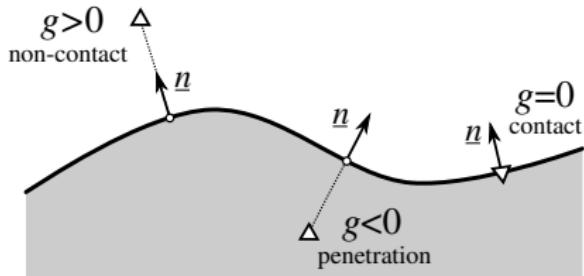
Gap function

■ Gap function g

- gap = - penetration
- asymmetric function
- defined for
 - separation $g > 0$
 - contact $g = 0$
 - penetration $g < 0$
- governs normal contact

■ Master and slave split

Gap function is determined for all slave points with respect to the master surface



Gap between a slave point and a master surface

Gap function

■ Gap function g

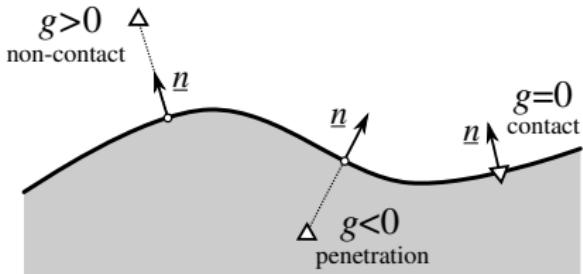
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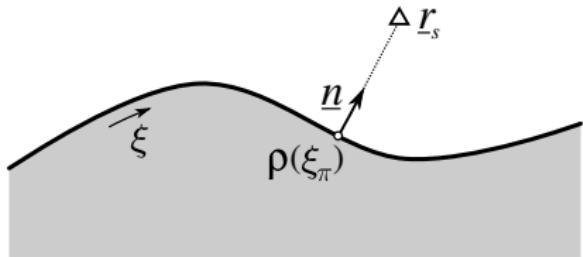
Gap function is determined for all slave points with respect to the master surface

■ Normal gap

$g_n = \underline{n} \cdot [\underline{r}_s - \rho(\xi_\pi)]$,
 \underline{n} is a unit normal vector, \underline{r}_s slave point, $\rho(\xi_\pi)$ projection point at master surface



Gap between a slave point and a master surface



Definition of the normal gap

Frictionless or normal contact conditions

- **No penetration**

Always non-negative gap

$$g \geq 0$$

- **No adhesion**

Always non-positive contact pressure

$$\underline{\sigma}_n^* \leq 0$$

- **Complementary condition**

*Either zero gap and non-zero pressure, or
non-zero gap and zero pressure*

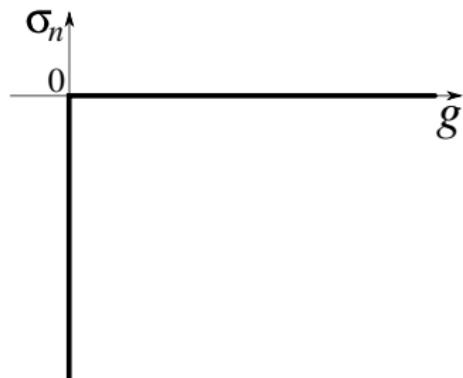
$$g \underline{\sigma}_n = 0$$

- **No shear transfer (automatically)**

$$\underline{\sigma}_t^{**} = 0$$

$$\underline{\sigma}_n^* = (\underline{\underline{\sigma}} \cdot \underline{n}) \cdot \underline{n} = \underline{\underline{\sigma}} : (\underline{n} \otimes \underline{n})$$

$$\underline{\sigma}_t^{**} = \underline{\underline{\sigma}} \cdot \underline{n} - \sigma_n \underline{n} = \underline{n} \cdot \underline{\underline{\sigma}} \cdot (\underline{\underline{I}} - \underline{n} \otimes \underline{n})$$



Scheme explaining normal contact conditions

Frictionless or normal contact conditions

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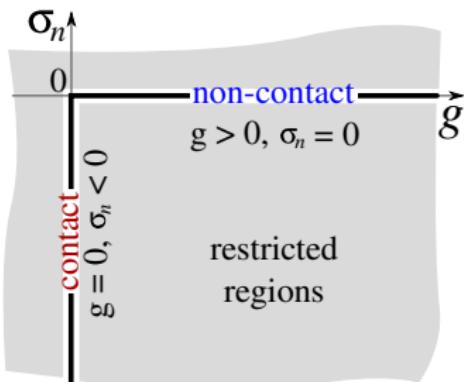
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Improved scheme explaining
normal contact conditions

Frictionless or normal contact conditions

In mechanics:

Normal contact conditions

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Frictionless contact conditions

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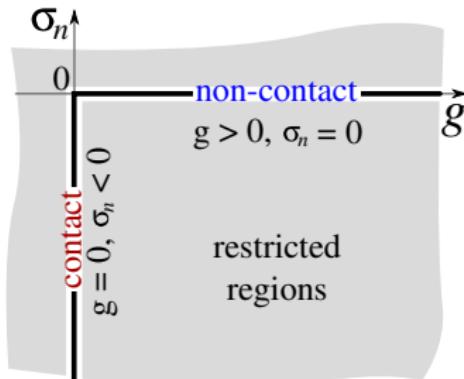
Hertz^[1]-Signorini^[2] conditions

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Hertz^[1]-Signorini^[2]-Moreau^[3] conditions

also known in **optimization theory** as

Karush^[4]-Kuhn^[5]-Tucker^[6] conditions



Improved scheme explaining
normal contact conditions

$$g \geq 0, \quad \sigma_n \leq 0, \quad g\sigma_n = 0$$

¹**Heinrich Rudolf Hertz** (1857–1894) a German physicist who first formulated and solved the frictionless contact problem between elastic ellipsoidal bodies.

²**Antonio Signorini** (1888–1963) an Italian mathematical physicist who gave a general and rigorous mathematical formulation of contact constraints.

³**Jean Jacques Moreau** (1923) a French mathematician who formulated a non-convex optimization problem based on these conditions and introduced pseudo-potentials in contact mechanics.

⁴**William Karush** (1917–1997), ⁵**Harold William Kuhn** (1925) American mathematicians,

⁶**Albert William Tucker** (1905–1995) a Canadian mathematician.

Contact problem

≈ Problem

Find such contact pressure

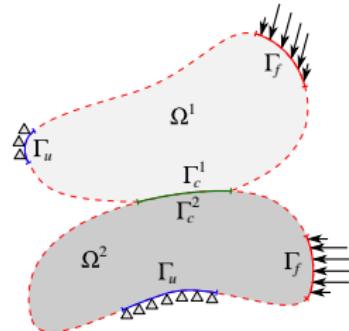
$$p = -\underline{n} \cdot \underline{\sigma} \cdot \underline{n} \geq 0$$

which being applied at Γ_c^1 and Γ_c^2 results in

$$\underline{x}^1 = \underline{x}^2, \forall \underline{x}^1 \in \Gamma_c^1, \underline{x}^2 \in \Gamma_c^2$$

and evidently

$$\Omega^1(t) \cap \Omega^2(t) = \emptyset$$



Two bodies in contact

- Unfortunately, we do not know Γ_c^1 in advance, it is also an unknown of the problem.

- **Related problem**

Suppose that we know p on Γ_c

Then what is the corresponding displacement field \underline{u} in Ω^i ?
Or the other way around?

Contact problem

≈ Problem

Find such contact pressure

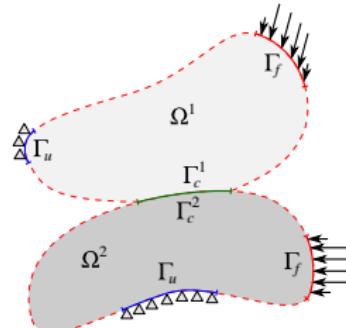
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See Flamant problem TD

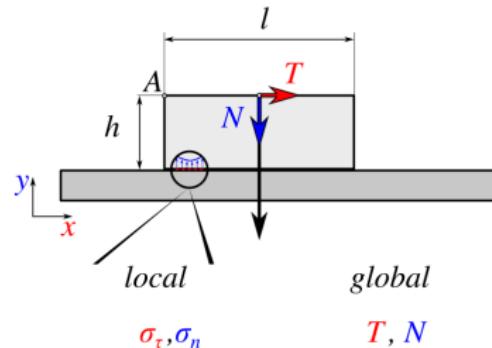
Evidence of friction

- Existence of frictional resistance is evident
- Independence of the nominal contact area

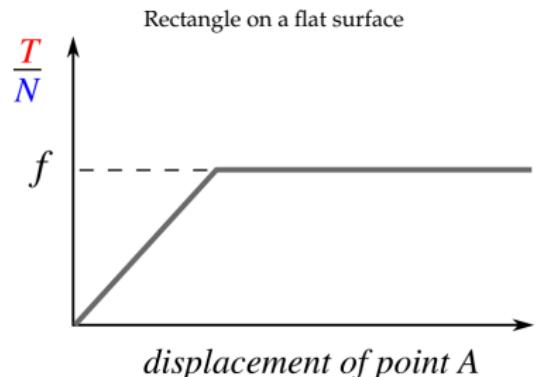


Think about adhesion and introduce a threshold in the interface τ_c

- Globally:
 - stick: $T < T_c(N)$
 - slip: $T = T_c(N)$
- From experiments:
 - Threshold $T_c \sim N$
 - Friction coefficient $f = |T_c/N|$
- Locally
 - stick: $\sigma_\tau < \tau_c(\sigma_n)$
 - slip: $\sigma_\tau = f\sigma_n$



$$f = \max(|\sigma_\tau/\sigma_n|) \quad f = \max(|T/N|)$$



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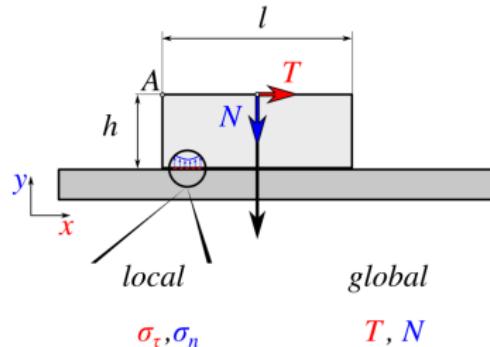
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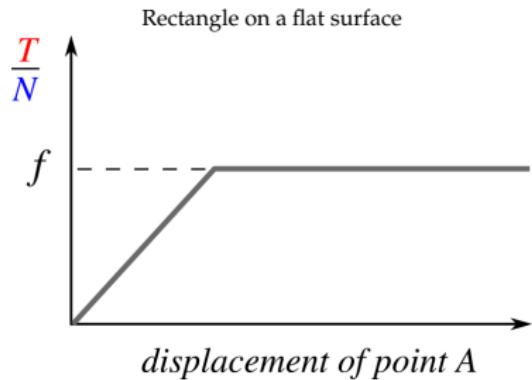


Torque



$$f = \max(|\sigma_\tau/\sigma_n|)$$

$$f = \max(|T/N|)$$

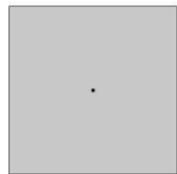
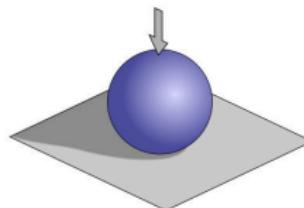
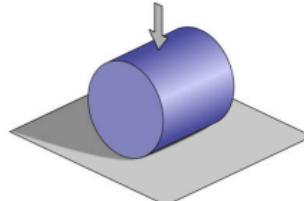
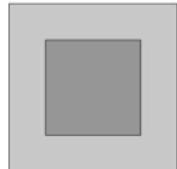
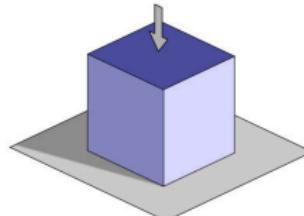


Types of contact

- Known contact zone
 - conformal geometry
flat-to-flat, cylinder in a hole
 - initially non-conformal geometry but huge pressure resulting in full contact
- Unknown contact zone
general case
- Point and line contact
- Frictionless
conservative, energy minimization problem
- Frictional
path-dependent solution, from the first touch to the current moment



Example

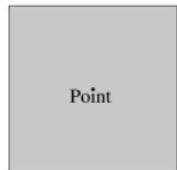
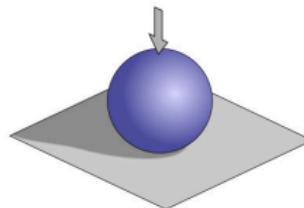
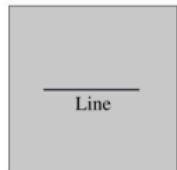
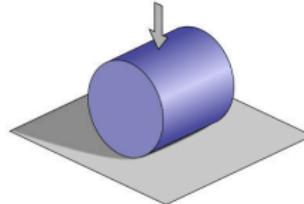
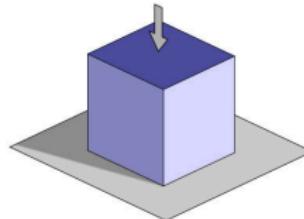


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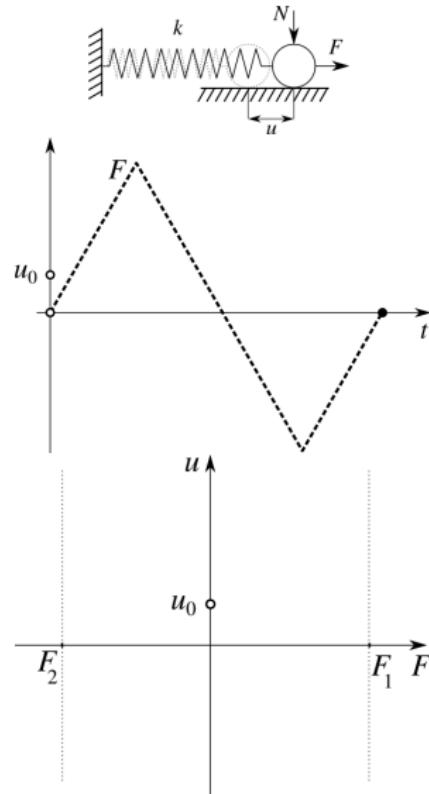


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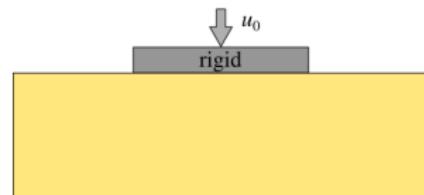
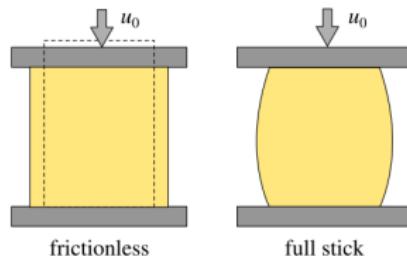
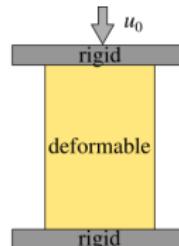
Example



Analogy with boundary conditions

Flat geometry

- Compression of a cylinder
- Frictionless $u_z = u_0$
- Full stick conditions $\underline{u} = u_0 \underline{e}_z$
- Rigid flat indenter $u_z = u_0$



Analogy with boundary conditions

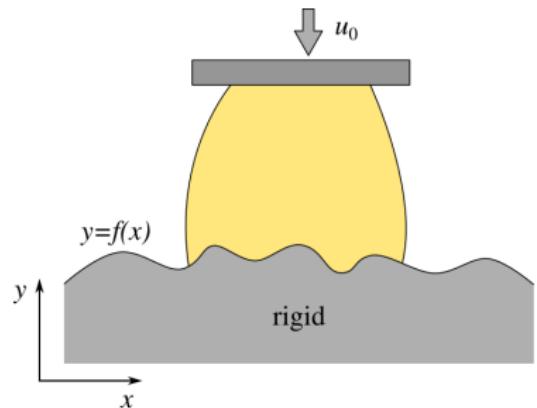
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Curved geometry

- Polar/spherical coordinates
 $\underline{u}_r = \underline{u}_0$
- If frictionless contact on rigid surface $y = f(x)$ is retained by high pressure

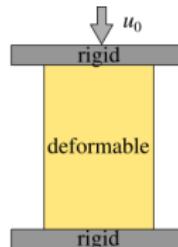
$$(\underline{X} + \underline{u}) \cdot \underline{e}_y = f((\underline{X} + \underline{u}) \cdot \underline{e}_x)$$



Analogy with boundary conditions

Flat geometry

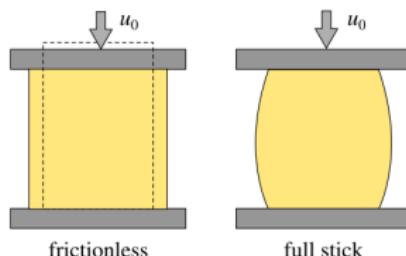
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Curved geometry

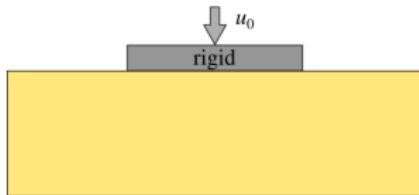
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Transition to finite friction

-  ≈ From full stick, decrease f by keeping $\underline{u}_z = 0$ and by replacing in-plane Dirichlet BC by in-plane Neumann BC



Analogy with boundary conditions II

In general

- Type I: prescribed tractions

$$p(x, y), \tau_x(x, y), \tau_y(x, y)$$



- Type II: prescribed displacements

$$\underline{u}(x, y)$$

- Type III: tractions and displacements

$$u_z(x, y), \tau_x(x, y), \tau_y(x, y) \text{ or}$$

$$p(x, y), u_x(x, y), u_y(x, y)$$

- Type IV: displacements and relation between tractions

$$u_z(x, y), \tau_x(x, y) = \pm fp(x, y)$$

④

To be continued...

Concentrated forces

- Normal force: in-plane stresses and displacements (plane strain)

$$\sigma_r = -\frac{2N}{\pi} \frac{\cos(\theta)}{r} \text{ or } \sigma_x = -\frac{2N}{\pi} \frac{x^2 y}{(x^2+y^2)^2}, \quad \sigma_y = -\frac{2N}{\pi} \frac{y^3}{(x^2+y^2)^2}, \quad \sigma_{xy} = -\frac{2N}{\pi} \frac{xy^2}{(x^2+y^2)^2}$$

$$u_r = \frac{1+\nu}{\pi E} N \cos(\theta) [2(1-\nu) \ln(r) - (1-2\nu)\theta \tan(\theta)] + C \cos(\theta)$$

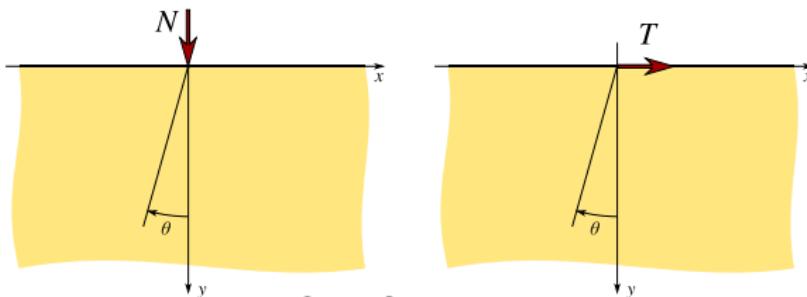
$$u_\theta = \frac{1+\nu}{\pi E} N \sin(\theta) [2(1-\nu) \ln(r) - 2\nu + (1-2\nu)(1-2\theta \tan(\theta))] - C \sin(\theta)$$

- Tangential force

$$\sigma_r = \frac{2T}{\pi} \frac{\sin(\theta)}{r} \text{ or } \sigma_x = -\frac{2T}{\pi} \frac{x^3}{(x^2+y^2)^2}, \quad \sigma_y = -\frac{2T}{\pi} \frac{xy^2}{(x^2+y^2)^2}, \quad \sigma_{xy} = -\frac{2T}{\pi} \frac{x^2 y}{(x^2+y^2)^2}$$

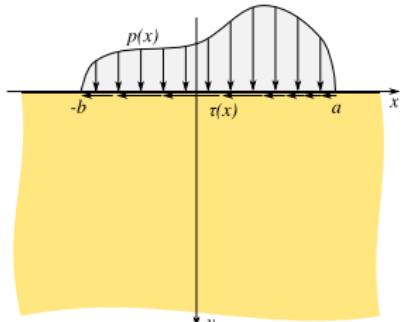
$$u_r = -\frac{1+\nu}{\pi E} T \sin(\theta) [2(1-\nu) \ln(r) - (1-2\nu)\theta \tan(\theta)] - C \sin(\theta)$$

$$u_\theta = \frac{1+\nu}{\pi E} T \cos(\theta) [2(1-\nu) \ln(r) - 2\nu + (1-2\nu)(1+2\theta \tan(\theta))] + C \cos(\theta)$$



Distributed load

- Distributed tractions $p(x)dx = dN(x)$,
 $\tau(x)dx = dT(x)$
- Use superposition principle for the stress state and for displacements



Tractions on the surface

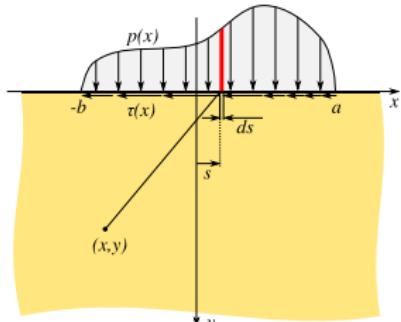
$$\sigma_x(x, y) = -\frac{2y}{\pi} \int_{-b}^a \frac{p(s)(x-s)^2 ds}{((x-s)^2 + y^2)^2} - \frac{2}{\pi} \int_{-b}^a \frac{\tau(s)(x-s)^3 ds}{((x-s)^2 + y^2)^2}$$

$$\sigma_y(x, y) = -\frac{2y^3}{\pi} \int_{-b}^a \frac{p(s) ds}{((x-s)^2 + y^2)^2} - \frac{2y^2}{\pi} \int_{-b}^a \frac{\tau(s)(x-s) ds}{((x-s)^2 + y^2)^2}$$

$$\sigma_{xy}(x, y) = -\frac{2y^2}{\pi} \int_{-b}^a \frac{p(s)(x-s) ds}{((x-s)^2 + y^2)^2} - \frac{2y}{\pi} \int_{-b}^a \frac{\tau(s)(x-s)^2 ds}{((x-s)^2 + y^2)^2}$$

Distributed load

- Distributed tractions $p(x)dx = dN(x)$,
 $\tau(x)dx = dT(x)$
- Use superposition principle for the stress state and for displacements



Tractions on the surface

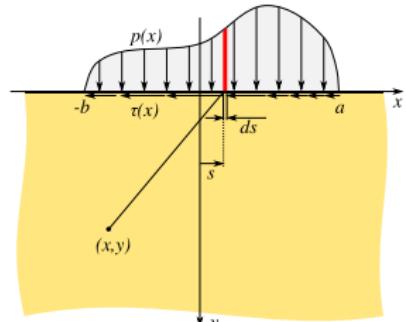
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Distributed load

- Distributed tractions $p(x)dx = dN(x)$,
 $\tau(x)dx = dT(x)$
- Use superposition principle for the stress state and for displacements
- Consider displacements on the surface

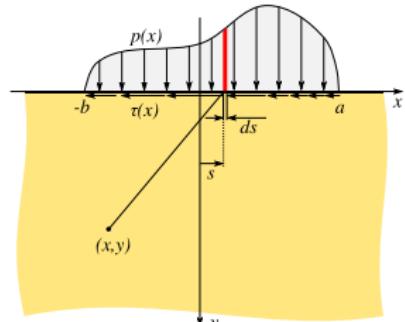


Tractions on the surface

$$u_x(x, 0) = -\frac{(1 - 2\nu)(1 + \nu)}{2E} \left[\int_{-b}^x p(s) ds - \int_x^a p(s) ds \right] - \frac{2(1 - \nu^2)}{\pi E} \int_{-b}^a \tau(s) \ln |x-s| ds + C_1$$

Distributed load

- Distributed tractions $p(x)dx = dN(x)$,
 $\tau(x)dx = dT(x)$
- Use superposition principle for the stress state and for displacements
- Consider displacements on the surface
- Or rather their derivatives along the surface



Tractions on the surface

$$u_x(x, 0) = -\frac{(1-2\nu)(1+\nu)}{2E} \left[\int_{-b}^x p(s) ds - \int_x^a p(s) ds \right] - \frac{2(1-\nu^2)}{\pi E} \int_{-b}^a \tau(s) \ln |x-s| ds + C_1$$

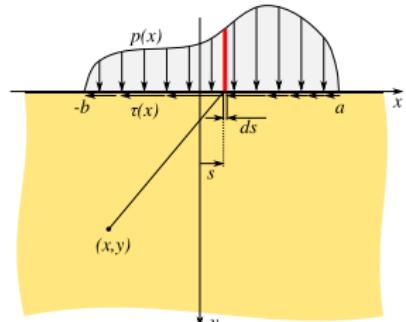
$$u_{x,x}(x, 0) = -\frac{(1-2\nu)(1+\nu)}{E} p(x) - \frac{2(1-\nu^2)}{\pi E} \int_{-b}^a \frac{\tau(s)}{x-s} ds$$



Near-surface stress state

Distributed load

- Distributed tractions $p(x)dx = dN(x)$,
 $\tau(x)dx = dT(x)$
- Use superposition principle for the stress state and for displacements
- Consider displacements on the surface
- Or rather their derivatives along the surface

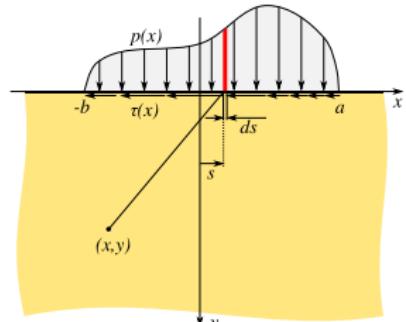


Tractions on the surface

$$u_y(x, 0) = \frac{(1 - 2\nu)(1 + \nu)}{2E} \left[\int_{-b}^x \tau(s) ds - \int_x^a \tau(s) ds \right] - \frac{2(1 - \nu^2)}{\pi E} \int_{-b}^a p(s) \ln |x-s| ds + C_2$$

Distributed load

- Distributed tractions $p(x)dx = dN(x)$,
 $\tau(x)dx = dT(x)$
- Use superposition principle for the stress state and for displacements
- Consider displacements on the surface
- Or rather their derivatives along the surface



Tractions on the surface

$$u_y(x, 0) = \frac{(1 - 2\nu)(1 + \nu)}{2E} \left[\int_{-b}^x \tau(s) ds - \int_x^a \tau(s) ds \right] - \frac{2(1 - \nu^2)}{\pi E} \int_{-b}^a p(s) \ln |x-s| ds + C_2$$

$$u_{y,x}(x, 0) = \frac{(1 - 2\nu)(1 + \nu)}{E} \tau(x) - \frac{2(1 - \nu^2)}{\pi E} \int_{-b}^a \frac{p(s)}{x-s} ds$$

Rigid stamp problem

- Link displacement derivatives with tractions

$$\int_{-b}^a \frac{\tau(s)}{x-s} ds = -\frac{\pi(1-2\nu)}{2(1-\nu)} p(x) - \frac{\pi E}{2(1-\nu^2)} u_{x,x}(x, 0)$$

$$\int_{-b}^a \frac{p(s)}{x-s} ds = \frac{\pi(1-2\nu)}{2(1-\nu)} \tau(x) - \frac{\pi E}{2(1-\nu^2)} u_{y,x}(x, 0)$$

- If in contact interface we can prescribe $p, u_{x,x}$ or $\tau, u_{y,x}$, then the problem reduces to

$$\int_{-b}^a \frac{\mathcal{F}(x)}{x-s} ds = \mathcal{U}(x)$$

- The general solution (case $a = b$):

$$\mathcal{F}(x) = \frac{1}{\pi^2 \sqrt{a^2 - x^2}} \int_{-a}^a \frac{\sqrt{a^2 - s^2} \mathcal{U}(s) ds}{x-s} + \frac{C}{\pi \sqrt{a^2 - x^2}}, \quad C = \int_{-a}^a \mathcal{F}(s) ds$$

Rigid stamp problem

- Link displacement derivatives with tractions

$$\int_{-b}^a \frac{\tau(s)}{x-s} ds = -\frac{\pi(1-2\nu)}{2(1-\nu)} p(x) - \frac{\pi E}{2(1-\nu^2)} u_{x,x}(x, 0)$$

$$\int_{-b}^a \frac{p(s)}{x-s} ds = \frac{\pi(1-2\nu)}{2(1-\nu)} \tau(x) - \frac{\pi E}{2(1-\nu^2)} u_{y,x}(x, 0)$$

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flat frictionless punch, consider P.V.

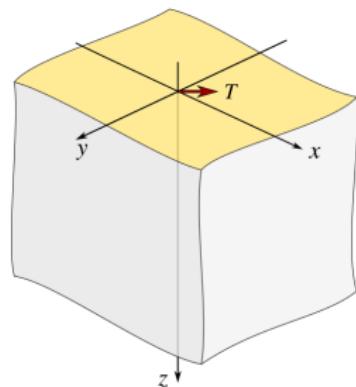
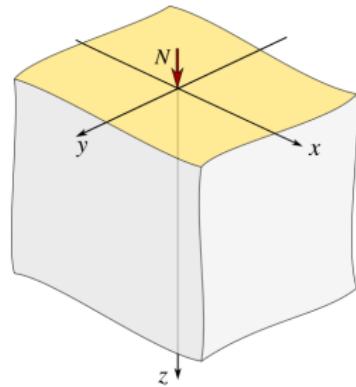
Three-dimensional problem

- Analogy to Flamant's problem
- Potential functions of Boussinesq
- Boussinesq problem
concentrated normal force
- Cerruti problem
concentrated tangential force
- Displacements decay as $\sim r^{-1}$

$$u_r(x, y, 0) = -\frac{1-2\nu}{4\pi G} \frac{N}{\sqrt{x^2 + y^2}}$$

$$u_z(x, y, 0) = \frac{1-\nu}{4\pi G} \frac{N}{\sqrt{x^2 + y^2}}$$

- Stress decay as $\sim r^{-2}$
- Superposition principle



Why is the sky dark at night?

Why is the sky dark at night?

- Olbers' paradox or “dark night sky paradox”
- Two nominally-flat elastic half-spaces in contact
- At small scale they are rough with asperity density D
- Vertical displacement decay $u_z \sim 1/r$
- At every asperity, force F
- Sum up displacements induced by all forces*

$$u_z \sim \int_0^{2\pi} \int_{r_0}^R \frac{F}{r} r dr d\phi \xrightarrow[R \rightarrow \infty]{} \infty$$

*In case of light intensity I , it decays as $1/r^2$ but the integral is in volume for a constant start density the integral light intensity is:

$$\mathcal{I} \sim \int_0^{2\pi} \int_0^{\pi/2} \int_{r_0}^R \frac{I}{r^2} \underbrace{r^2 \sin(\theta) dr d\phi d\theta}_{\text{Volume element}} \xrightarrow[R \rightarrow \infty]{} \infty$$

Classical contact problems

- Various problems with rigid flat stamps:
circular, elliptic, frictionless, full-stick, finite friction

- Hertz theory

normal frictionless contact of elastic solids

$$E_i, \nu_i \text{ and } z_i = A_i x^2 + B_i y^2 + C_i xy, \quad i = 1, 2$$



- Wedges (*coin*) and cones

- Circular inclusion in a conforming hole

Steuermann, 1939, Goodman, Keer, 1965

- Frictional indentation $z \sim x^n$

Incremental approach Mossakovski, 1954

self-similar solution Spence, 1968, 1975

- Adhesive contact Johnson et al, 1971, 1976

- Contact with layered materials (coatings)

- Elastic-plastic and viscoelastic materials

- Sliding/rolling of non-conforming bodies

Cattaneo (1938), Mindlin (1949), Galin (1953), Goryacheva (1998)

Note: $u_r \sim (1 - 2\nu)/G$, so if $(1 - 2\nu_1)/G_1 = (1 - 2\nu_2)/G_2$ tangential tractions do not change normal ones



K.L. Johnson (1985) J.R. Barber (2018)



I.G. Goryacheva (1998)



V.L. Popov (2017, 2nd Ed.)

Hertzian contact

- No friction, no adhesion
- Two elastic materials
 E_1, ν_1, E_2, ν_2
- Effective elastic modulus

$$\frac{1}{E^*} = \frac{1 - \nu_1^2}{E_1} + \frac{1 - \nu_2^2}{E_2}$$

- Two parabolic surfaces
 $z_1 = ax_1^2 + by_1^2, z_2 = cx_2^2 + dy_2^2$
- Solids of revolution or cylinders R_1, R_2 ,
effective curvature radius:

$$\frac{1}{R^*} = \frac{1}{R_1} + \frac{1}{R_2}$$

- Displacement and contact radius (only for 3D!):

$$\delta = a^2/R^*$$

- Contact pressure:

$$p(r) = p_0 \sqrt{1 - \frac{r^2}{a^2}}, |r| < a$$

156

Ueber die Berührung fester elastischer Körper.

(Von Herrn Heinrich Hertz.)

In der Theorie der Elastizität werden als Ursachen der Deformationen theils Kräfte, welche auf das Innere der Körper wirken, theils auf die Oberfläche wirkende Druckkräfte angenommen. Für beide Arten von Kräften kann der Fall eintreten, dass dieselben in einzelnen unendlich kleinen Theilen der Körper unendlich gross werden, so zwar, dass die Integrale der Kräfte über diese Theile genommen einen endlichen Werth behalten. Beschreiben wir alsdann um den Unstetigkeitspunkt eine geschlossene Fläche, deren Dimensionen sehr klein gegen die Dimensionen des ganzen Körpers sind, sehr gross hängen im Vergleich zu den Dimensionen des Theils, in welchem die Kräfte angreifen, so können die Deformationen innerhalb und innerhalb dieser Fläche ganz unabhängig voneinander betrachtet werden. Außerhalb hingegen der Deformationen ab von der Gestalt des Gesamtkörpers, der Verteilung der übrigen Kräfte und den endlichen Integralen der Kraftkomponenten im Unstetigkeitspunkte, innerhalb hängen sie nur ab von der Verteilung der im Innern selbst angreifenden Kräfte. Die Drücke und Deformationen im Innern sind gegen die im Aussenraum unendlich gross.

Im Folgenden wollen wir einen hierher gehörigen Fall behandeln, der praktisches Interesse hat*, den Fall nämlich, dass zwei elastische isotrope Körper sich in einem sehr kleinen Theil ihrer Oberfläche berühren, und durch diesen Theil einen endlichen Druck der eine auf den andern ausüben. Die sich berührenden Oberflächen stellen wir uns als vollkommen glatt vor, d. h. wir nehmen nur einen senkrechten Druck zwischen den sich berührenden Theilen an. Das beiden Körper nach der Deformation gemeinsame Stück der Oberfläche wollen wir die Druckfläche, die Begrenzung

* Vgl. Wiede, Die Lehre von der Elastizität und Festigkeit, Frag 1867; I, p. 45. Grashof, Theorie der Elastizität und Festigkeit, Berlin 1878; p. 49–54.

Original paper by Henrich Hertz "On the contact of elastic solids" (ENG trans.) (16 pages)

"His theory, worked out during the Christmas vacation 1880 at the age of 23(!), aroused considerable interest . . ." K.L. Johnson

Hertzian contact

- No friction, no adhesion

- Two elastic materials

$$E_1, \nu_1, E_2, \nu_2$$

- Effective elastic modulus

$$\frac{1}{E^*} = \frac{1 - \nu_1^2}{E_1} + \frac{1 - \nu_2^2}{E_2}$$

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$$z_1 = ax_1^2 + by_1^2, z_2 = cx_2^2 + dy_2^2$$

- Solids of revolution or cylinders R_1, R_2 ,
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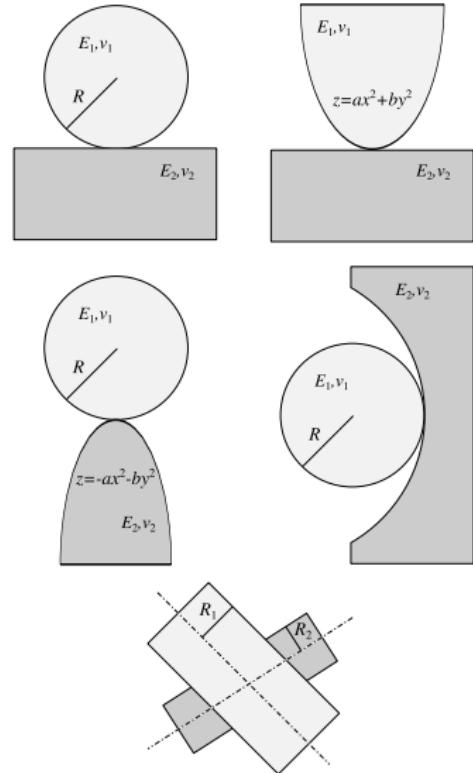
$$\frac{1}{R^*} = \frac{1}{R_1} + \frac{1}{R_2}$$

- Displacement and contact radius (only
for 3D!):

$$\delta = a^2/R^*$$

- Contact pressure:

$$p(r) = p_0 \sqrt{1 - \frac{r^2}{a^2}}, |r| < a$$



Geometries resolved in the framework of Hertz theory

Hertzian contact

- No friction, no adhesion

- Two elastic materials

$$E_1, \nu_1, E_2, \nu_2$$

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- Contact pressure:

$$p(r) = p_0 \sqrt{1 - \frac{r^2}{a^2}}, |r| < a$$

- Line contact (cylinders):

$$a = \left(\frac{4PR^*}{\pi E^*} \right)^{1/2}$$

$$p_0 = \frac{2P}{\pi a}$$

- Solids of revolution:

$$a = \left(\frac{3PR^*}{4E^*} \right)^{1/3}$$

$$p_0 = \frac{3P}{2\pi a^2}$$



Thank you for your attention!