

Contact mechanics and elements of tribology

Lecture 6.

Computational Contact Mechanics

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Centre des Matériaux, Evry, France*

@ Centre des Matériaux (virtually)
February 10, 2021

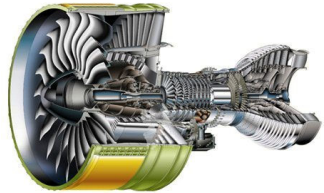


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- Introduction
- Governing equations
- Optimization methods
- Resolution algorithm
- Examples

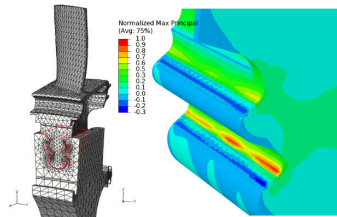
Industrial and natural contact problems

1 Assembled parts, e.g. engines



Aircraft's engine GP 7200

www.safran-group.com



[1] M. W. R. Savage

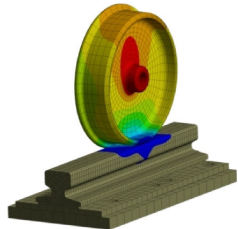
J. Eng. Gas Turb. Power, 134:012501 (2012)

Industrial and natural contact problems

- 1 Assembled parts, e.g. engines
- 2 Railroad contacts



High speed train TGV www.sncf.com



Wilde/ANSYS wildeanalysis.co.uk

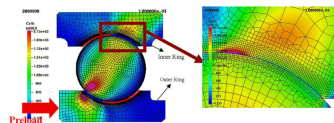
Industrial and natural contact problems

- 1 Assembled parts, e.g. engines
- 2 Railroad contacts
- 3 Gears and bearings



Bearings

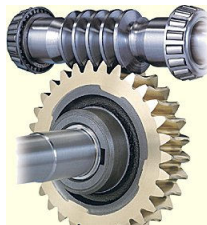
www.skf.com



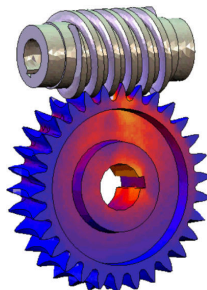
[1] F. Massi, J. Rocchi, A. Culla, Y. Berthier
Mech. Syst. Signal Pr., 24:1068-1080 (2010)

Industrial and natural contact problems

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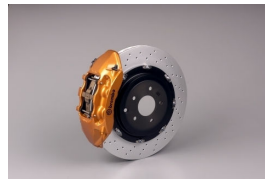
Helical gear www.tpg.com.tw



www.mscsoftware.com

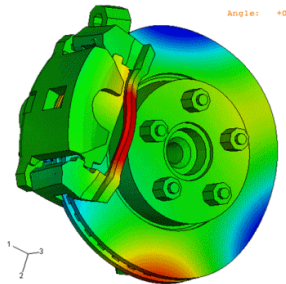
Industrial and natural contact problems

- 1 Assembled parts, e.g. engines
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- 3 Gears and bearings
- 4 Breaking systems



Assembled breaking system

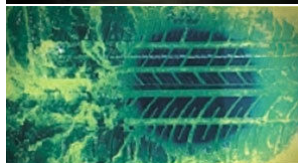
www.brembo.com



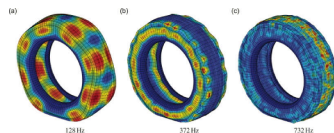
www.mechanicalengineeringblog.com

Industrial and natural contact problems

- 1 Assembled parts, e.g. engines
- 2 Railroad contacts
- 3 Gears and bearings
- 4 Breaking systems
- 5 Tire-road contact



Tire-road contact www.michelin.com



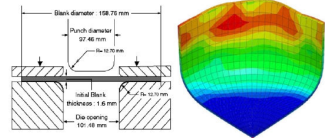
[1] M. Brinkmeier, U. Nackenhorst, S. Petersen, O. von Estorff, J. Sound Vib., 309:20-39 (2008)

Industrial and natural contact problems

- 1 Assembled parts, e.g. engines
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- 4 Breaking systems
- 5 Tire-road contact
- 6 Metal forming



Deep drawing www.thomasnet.com



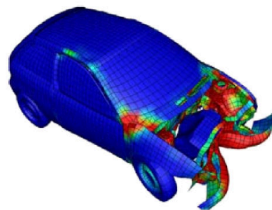
[1] G. Rousselier, F. Barlat, J. W. Yoon
Int. J. Plasticity, 25:2383-2409 (2009)

Industrial and natural contact problems

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- 4 Breaking systems
- 5 Tire-road contact
- 6 Metal forming
- 7 Crash tests



Crash-test www.porsche.com



[1] O. Klyavin, A. Michailov, A. Borovkov
www.fea.ru

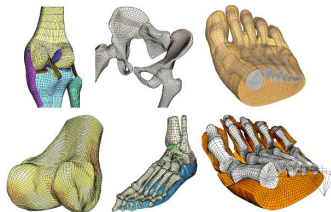
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- 7 Crash tests
- 8 Biomechanics



Human articulations

www.sportssupplements.net



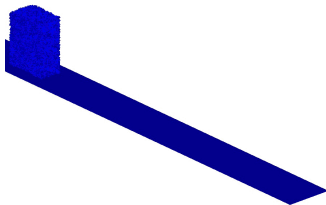
*J. A. Weiss, University of Utah
Musculoskeletal Research Laboratories*

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- 8 Biomechanics
- 9 Granular materials



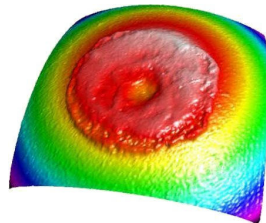
Sand dunes www.en.wikipedia.org



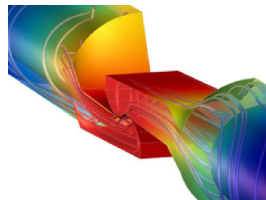
E. Azema et al, LMGC90

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- 9 Granular materials
- 10 Electric contacts



Damage at electric contact zone
www.taicaan.com



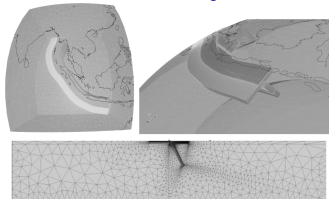
Simulation of electric current
www.comsol.com

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- 7 Crash tests
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- 11 Tectonic motions



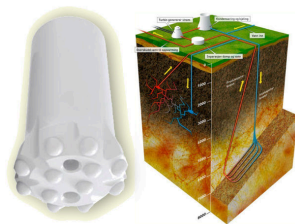
San-Andreas fault, by M. Rightmire
www.sciencedude.ocregister.com



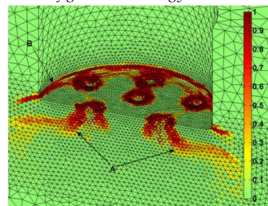
[1] J.D. Garaud, L. Fleitout, G. Caillaud
Colloque CSMA (2009)

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- 9 Granular materials
- 10 Electric contacts
- 11 Tectonic motions
- 12 Deep drilling



Drill Bit tool [RobitRocktools](#);
extraction of geothermal energy ([SINTEF, NTNU](#))



[1] T. Saksala, *Int. J. Numer. Anal. Meth. Geomech.* (2012)

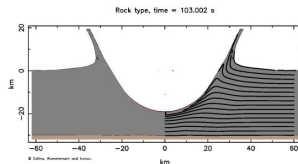
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- 12 Deep drilling
- 13 Impact and fragmentation



Impact crater, Arizona

www.MrEclipse.com et maps.google.com



Simulation of formation of Copernicus crater
Yue Z., Johnson B. C., et al. Projectile

remnants in central peaks of lunar impact
craters. Nature Geo 6 (2013)

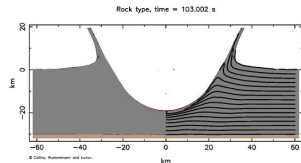
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- 14 etc.



Impact crater, Arizona

www.MrEclipse.com et maps.google.com



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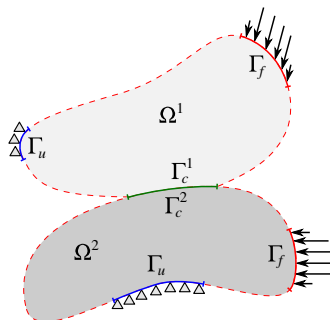
Equilibrium and contact conditions

■ Balance of momentum

$$\begin{cases} \nabla \cdot \underline{\underline{\sigma}} + \underline{\underline{f}}_v = 0 & \text{in } \Omega_{1,2} \\ \underline{\underline{\sigma}} \cdot \underline{n} = \underline{t}_0 & \text{on } \Gamma_f \\ \underline{u} = \underline{u}_0 & \text{on } \Gamma_u \\ ? & \text{on } \Gamma_c \end{cases}$$

■ Frictionless contact conditions (*intuitive*)

- 1 No penetration
- 2 No adhesion
- 3 No shear transfer



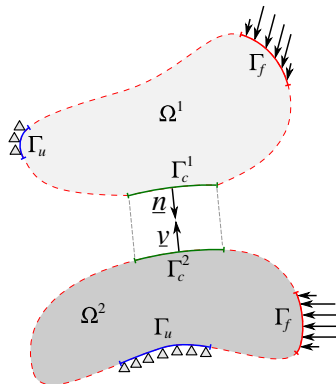
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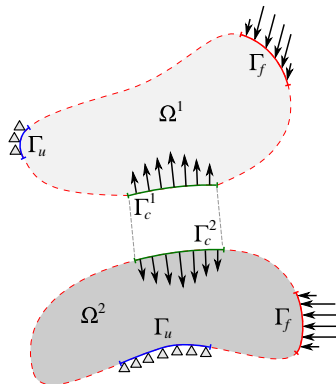
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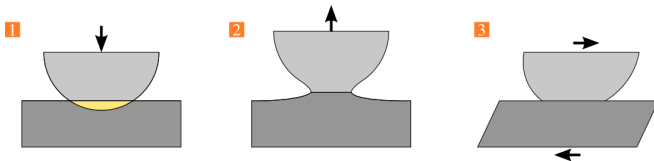
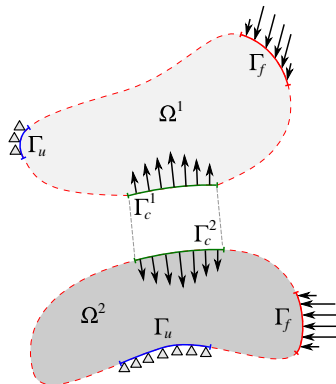
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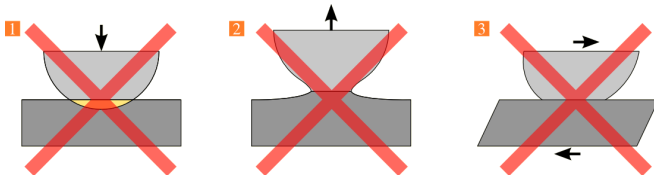
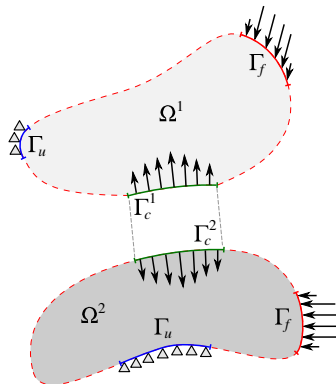
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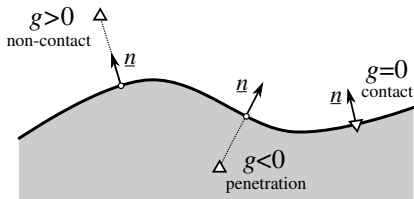


■ Gap function g

- gap = - penetration
- asymmetric function
- defined for
 - separation $g > 0$
 - contact $g = 0$
 - penetration $g < 0$
- governs normal contact

■ Master and slave split

Gap function is determined for all slave points with respect to the master surface

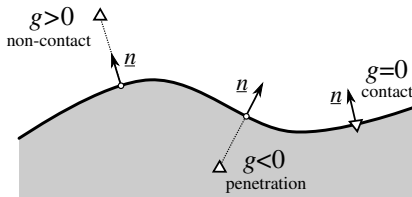


Gap between a slave point and a master surface

Gap function

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Gap between a slave point and a master surface

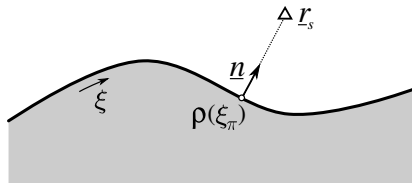
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■ Normal gap

$$g_n = \underline{n} \cdot [\underline{r}_s - \underline{\rho}(\xi_\pi)],$$

$\underline{n}$ is a unit normal vector, $\underline{r}_s$ slave point, $\underline{\rho}(\xi_\pi)$ projection point at master surface



Definition of the normal gap

Consider existence and uniqueness



Frictionless or normal contact conditions

■ No penetration

Always non-negative gap

$$g \geq 0$$

■ No adhesion

Always non-positive contact pressure

$$\sigma_n^* \leq 0$$

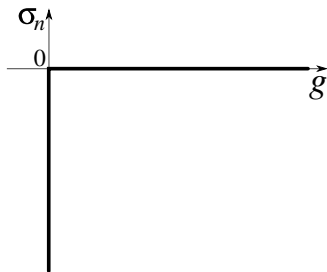
■ Complementary condition

Either zero gap and non-zero pressure, or non-zero gap and zero pressure

$$g \sigma_n = 0$$

■ No shear transfer (automatically)

$$\underline{\sigma}_t^{**} = 0$$



Scheme explaining normal contact conditions

$$\sigma_n^* = (\underline{\underline{\sigma}} \cdot \underline{n}) \cdot \underline{n} = \underline{\underline{\sigma}} : (\underline{n} \otimes \underline{n})$$

$$\underline{\sigma}_t^{**} = \underline{\underline{\sigma}} \cdot \underline{n} - \sigma_n \underline{n} = \underline{n} \cdot \underline{\underline{\sigma}} \cdot (\underline{I} - \underline{n} \otimes \underline{n})$$

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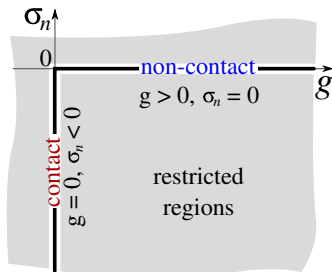
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Improved scheme explaining normal contact conditions

Frictionless or normal contact conditions

In mechanics:

Normal contact conditions

$\equiv$

Frictionless contact conditions

$\equiv$

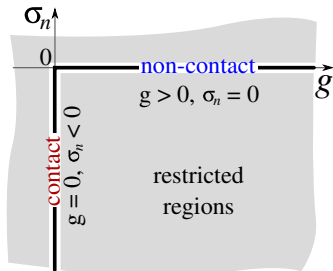
Hertz¹-Signorini^[2] conditions

$\equiv$

Hertz¹-Signorini^[2]-Moreau^[3] conditions

also known in **optimization theory** as

Karush^[4]-Kuhn^[5]-Tucker^[6] conditions



Improved scheme explaining normal contact conditions

$$g \geq 0, \quad \sigma_n \leq 0, \quad g\sigma_n = 0$$

¹Heinrich Rudolf Hertz (1857–1894) a German physicist who first formulated and solved the frictionless contact problem between elastic ellipsoidal bodies.

²Antonio Signorini (1888–1963) an Italian mathematical physicist who gave a general and rigorous mathematical formulation of contact constraints.

³Jean Jacques Moreau (1923) a French mathematician who formulated a non-convex optimization problem based on these conditions and introduced pseudo-potentials in contact mechanics.

⁴William Karush (1917–1997), ⁵Harold William Kuhn (1925) American mathematicians,

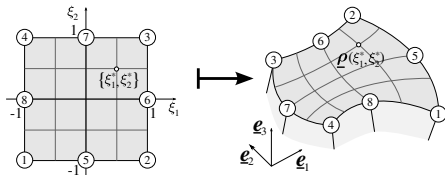
⁶Albert William Tucker (1905–1995) a Canadian mathematician.

Relative sliding

Recall:

- Convective coordinate in parent space $\xi_i \in (-1; 1)$
- Mapping to real space

$$\underline{\rho}(\xi_1, \xi_2, t) = \sum_{i=1}^8 N^i(\xi_1, \xi_2) \underline{\rho}^i(t)$$

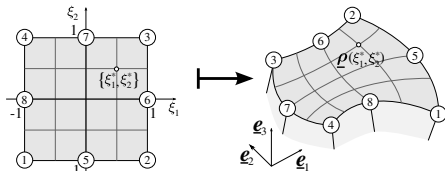


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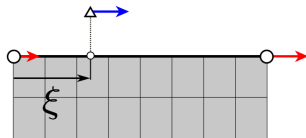
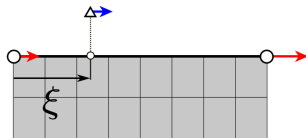
■ Tangential slip velocity $\underline{v}_t$

must take into account:

- only tangential component
- relative rigid body motion
- master's deformation

$$\underline{v}_t = \frac{\partial \underline{\rho}}{\partial \xi_1} \dot{\xi}_1 + \frac{\partial \underline{\rho}}{\partial \xi_2} \dot{\xi}_2$$

where $\partial \underline{\rho} / \partial \xi_i$ are the tangent vectors of the local basis and ξ_i are the convective coordinates.



Relative slip between a slave point and a deformable master surface

Relative sliding: example

Consider a one-dimensional example:

P is a projection of A on segment BC .

$$x_P = \xi x_C + (1 - \xi)x_B \quad (1)$$

Velocity of the projection point

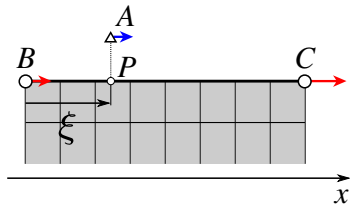
$$\dot{x}_P = \underbrace{\xi \dot{x}_C + (1 - \xi) \dot{x}_B}_{\frac{\partial x_P}{\partial t}} + \underbrace{(x_C - x_B) \dot{\xi}}_{\frac{\partial x_P}{\partial \xi} \dot{\xi}}$$

Subtract the velocity of point x_P for fixed ξ

$$v_t = \dot{x}_P - \frac{\partial x_P}{\partial t} = (x_C - x_B) \dot{\xi} = \frac{\partial x}{\partial \xi} \dot{\xi}$$

Compute tangential slip increment

$$\Delta g_t^{n+1} = \left. \frac{\partial x}{\partial \xi} \right|_{\xi^n} (\xi^{n+1} - \xi^n)$$



Example of a one-dimensional relative slip

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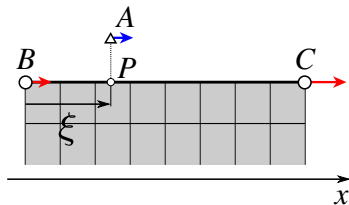
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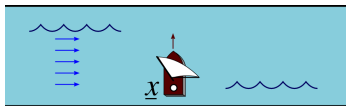
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Example of a one-dimensional relative slip



Ship-river analogy

Relative sliding: example

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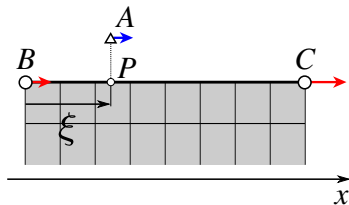
$$\dot{x}_P = \underbrace{\xi \dot{x}_C + (1 - \xi) \dot{x}_B}_{\frac{\partial x_P}{\partial t}} + \underbrace{(x_C - x_B) \dot{\xi}}_{\frac{\partial x_P}{\partial \xi} \dot{\xi}}$$

Subtract the velocity of point x_P for fixed ξ

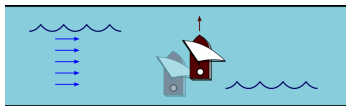
$$v_t = \dot{x}_P - \frac{\partial x_P}{\partial t} = (x_C - x_B) \dot{\xi} = \frac{\partial x}{\partial \xi} \dot{\xi}$$

Compute tangential slip increment

$$\Delta g_t^{n+1} = \left. \frac{\partial x}{\partial \xi} \right|_{\xi^n} (\xi^{n+1} - \xi^n)$$



Example of a one-dimensional relative slip



Ship-river analogy

Relative sliding: example

Consider a one-dimensional example:

P is a projection of A on segment BC .

$$x_P = \xi x_C + (1 - \xi)x_B \quad (1)$$

Velocity of the projection point

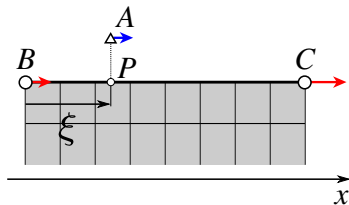
$$\dot{x}_P = \underbrace{\xi \dot{x}_C + (1 - \xi) \dot{x}_B}_{\frac{\partial x_P}{\partial t}} + \underbrace{(x_C - x_B) \dot{\xi}}_{\frac{\partial x_P}{\partial \xi} \dot{\xi}}$$

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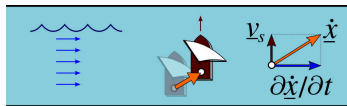
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Example of a one-dimensional relative slip



Ship-river analogy

Li derivative: the change of a vector field along the change of another vector field

Amontons-Coulomb's friction

- **No contact** $g > 0, \sigma_n = 0$

- **Stick** $|\underline{v}_t| = 0$

Inside slip surface/Coulomb's cone

$$f = |\underline{\sigma}_t| - \mu|\sigma_n| < 0$$

- **Slip** $|\underline{v}_t| > 0$

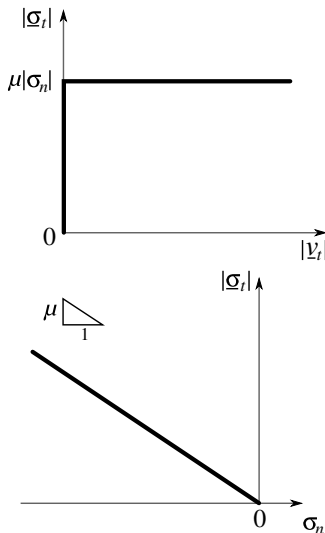
On slip surface/Coulomb's cone

$$f = |\underline{\sigma}_t| - \mu|\sigma_n| = 0$$

- **Complementary condition**

Either zero velocity and negative slip criterion, or non-zero velocity and zero slip criterion

$$|\underline{v}_t| \left(|\underline{\sigma}_t| - \mu|\sigma_n| \right) = 0$$



Scheme explaining frictional contact conditions

Amontons-Coulomb's friction

- **No contact** $g > 0, \sigma_n = 0$

- **Stick** $|\underline{v}_t| = 0$

Inside slip surface/Coulomb's cone

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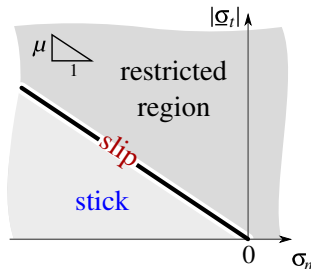
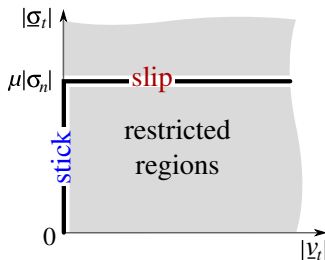
On slip surface/Coulomb's cone

$$f = |\underline{\sigma}_t| - \mu|\sigma_n| = 0$$

- **Complementary condition**

Either zero velocity and negative slip criterion, or non-zero velocity and zero slip criterion

$$|\underline{v}_t| \left(|\underline{\sigma}_t| - \mu|\sigma_n| \right) = 0$$



Improved scheme explaining frictional contact conditions

Amontons-Coulomb's friction

- **No contact** $g > 0$, $\sigma_n = 0$

- **Stick** $|\underline{v}_t| = 0$

Inside slip surface/Coulomb's cone

$$f = |\underline{\sigma}_t| - \mu|\sigma_n| < 0$$

- **Slip** $|\underline{v}_t| > 0$

On slip surface/Coulomb's cone

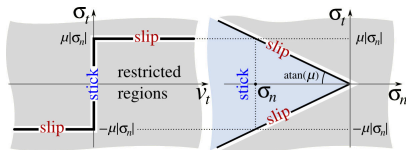
$$f = |\underline{\sigma}_t| - \mu|\sigma_n| = 0$$

- **Complementary condition**

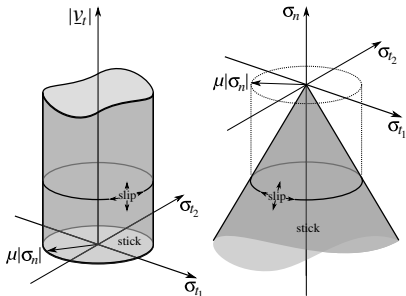
Either zero velocity and negative slip criterion, or non-zero velocity and zero slip criterion

$$|\underline{v}_t| \left(|\underline{\sigma}_t| - \mu|\sigma_n| \right) = 0$$

$$|\underline{v}_t| \geq 0, \quad |\underline{\sigma}_t| - \mu|\sigma_n| \leq 0, \quad |\underline{v}_t| \left(|\underline{\sigma}_t| - \mu|\sigma_n| \right) = 0$$



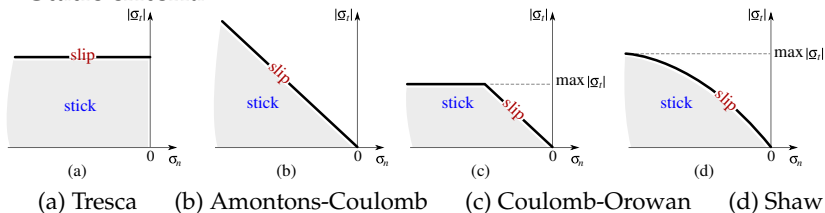
Scheme of 2D frictional contact



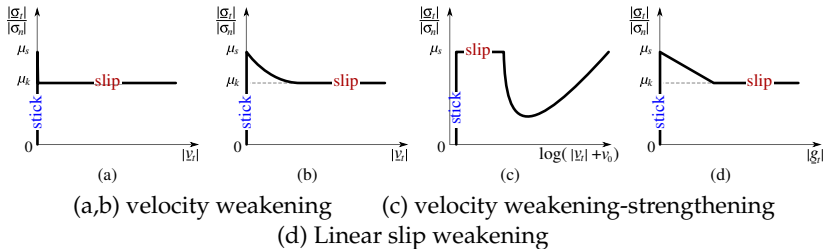
Scheme of 3D frictional contact

More friction laws

• Static criteria



• Kinetic criteria



- μ_s static and μ_k kinetic coefficients of friction.

Rate and state friction and regularization

- **Rate and state friction law**

- Rate $v_t = |\underline{v}_t|$ – relative slip velocity
- State θ – $\approx$ internal time
- Dieterich–Ruina–Perrin (1979, 83, 95)

Frictional resistance

$$\sigma_t^c = |\sigma_n| [\mu_s + b\theta + a \ln(v_t/v_0)]$$

Evolution of the state variable

$$\dot{\theta} = -\frac{v_t}{L} \left[\theta + \ln\left(\frac{v_t}{v_0}\right) \right]$$

Rate and state friction law

- **Prakash-Clifton friction law (1992,2000)**

- Viscous type evolution of frictional resistance σ_t
- $\dot{\sigma}_t = -\frac{v_t}{L} (\sigma_t + \mu\sigma_n)$

Prakash-Clifton regularization

Rate and state friction and regularization

- **Rate and state friction law**

-

-

Rate and state friction and regularization

- **Rate and state friction law**

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Prakash-Clifton regularization

Rate and state friction and regularization

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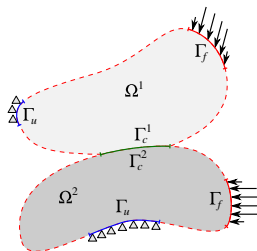
Rate and state friction and regularization

- Prakash-Clifton friction law (1992,2000)

From strong to weak form

- Balance of momentum and boundary conditions

$$\nabla \cdot \underline{\underline{\sigma}} + \underline{\underline{f}}_v = 0 \quad \text{in } \Omega = \Omega_1 \cup \Omega_2 \quad + \text{B.C.}$$



Two solids in contact

From strong to weak form

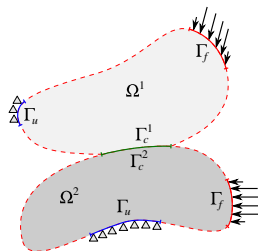
- Balance of momentum and boundary conditions

$$\nabla \cdot \underline{\underline{\sigma}} + \underline{\underline{f}}_v = 0 \quad \text{in } \Omega = \Omega_1 \cup \Omega_2 \quad + \text{B.C.}$$

- Balance of virtual works



$$\boxed{\int_{\partial\Omega} \underline{\underline{n}} \cdot \underline{\underline{\sigma}} \cdot \delta \underline{\underline{u}} d\Gamma} + \int_{\Omega} [\underline{\underline{f}}_v \cdot \delta \underline{\underline{u}} - \underline{\underline{\sigma}} \cdot \delta \nabla \underline{\underline{u}}] d\Omega = 0$$



Two solids in contact

From strong to weak form

- Balance of momentum and boundary conditions

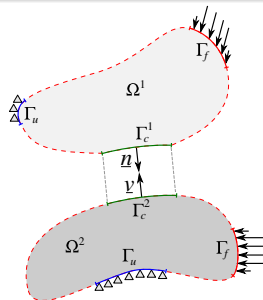
$$\nabla \cdot \underline{\underline{\sigma}} + \underline{\underline{f}}_v = 0 \quad \text{in } \Omega = \Omega_1 \cup \Omega_2 \quad + \text{B.C.}$$

- Balance of virtual works



$$\int_{\partial\Omega} \underline{\underline{n}} \cdot \underline{\underline{\sigma}} \cdot \delta \underline{\underline{u}} d\Gamma =$$

$$\int_{\bar{\Gamma}_c^1} \underline{\underline{n}} \cdot \underline{\underline{\sigma}} \cdot \delta \underline{\underline{\rho}} d\bar{\Gamma}_c^1 + \int_{\bar{\Gamma}_c^2} \underline{\underline{v}} \cdot \underline{\underline{\sigma}} \cdot \delta \underline{\underline{r}} d\bar{\Gamma}_c^2 + \int_{\Gamma_f} \underline{\underline{\sigma}}_0 \cdot \delta \underline{\underline{u}} d\Gamma_f$$



Two solids in contact

From strong to weak form

- Balance of momentum and boundary conditions

$$\nabla \cdot \underline{\underline{\sigma}} + \underline{\underline{f}}_v = 0 \quad \text{in } \Omega = \Omega_1 \cup \Omega_2 \quad + \text{B.C.}$$

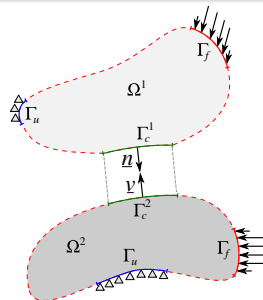
- Balance of virtual works



$$\int_{\partial\Omega} \underline{n} \cdot \underline{\underline{\sigma}} \cdot \delta \underline{u} \, d\Gamma \Rightarrow$$

$$\int_{\bar{\Gamma}_\ell^1} \underline{n} \cdot \underline{\underline{\sigma}} \cdot \delta \underline{\rho} \, d\bar{\Gamma}_c^1 + \int_{\bar{\Gamma}_\ell^2} \underline{v} \cdot \underline{\underline{\sigma}} \cdot \delta \underline{r} \, d\bar{\Gamma}_c^2 =$$

$$= \int_{\bar{\Gamma}_\ell^1} \underline{n} \cdot \underline{\underline{\sigma}} \cdot \delta(\underline{\rho} - \underline{r}) \, d\bar{\Gamma}_c^1 = \int_{\bar{\Gamma}_\ell^1} \left(\sigma_n \delta g_n + \underline{\underline{\sigma}}_t^T \delta \underline{\xi} \right) d\bar{\Gamma}_c^1$$



Two solids in contact

From strong to weak form

- Balance of momentum and boundary conditions

$$\nabla \cdot \underline{\underline{\sigma}} + \underline{\underline{f}}_v = 0 \quad \text{in } \Omega = \Omega_1 \cup \Omega_2 \quad + \text{B.C.}$$

- Balance of virtual works



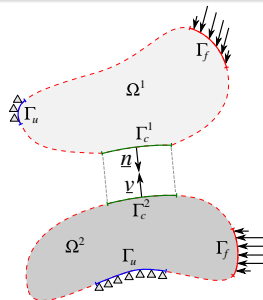
$$\int_{\partial\Omega} \underline{\underline{n}} \cdot \underline{\underline{\sigma}} \cdot \delta \underline{\underline{u}} d\Gamma \Rightarrow$$

$$\int_{\bar{\Gamma}_c^1} \underline{\underline{n}} \cdot \underline{\underline{\sigma}} \cdot \delta \underline{\underline{\rho}} d\bar{\Gamma}_c^1 + \int_{\bar{\Gamma}_c^2} \underline{\underline{v}} \cdot \underline{\underline{\sigma}} \cdot \delta \underline{\underline{r}} d\bar{\Gamma}_c^2 =$$

$$= \int_{\bar{\Gamma}_c^1} \underline{\underline{n}} \cdot \underline{\underline{\sigma}} \cdot \delta(\underline{\underline{\rho}} - \underline{\underline{r}}) d\bar{\Gamma}_c^1 = \int_{\bar{\Gamma}_c^1} \left(\sigma_n \delta g_n + \underline{\underline{\sigma}}_t^T \delta \underline{\underline{\xi}} \right) d\bar{\Gamma}_c^1$$

$$\int_{\Omega} \underline{\underline{\sigma}} \cdot \delta \nabla \underline{\underline{u}} d\Omega + \underbrace{\int_{\bar{\Gamma}_c^1} \left(\sigma_n \delta g_n + \underline{\underline{\sigma}}_t^T \delta \underline{\underline{\xi}} \right) d\bar{\Gamma}_c^1}_{\text{Contact term}} = \int_{\Gamma_f} \underline{\underline{\sigma}}_0 \cdot \delta \underline{\underline{u}} d\Gamma + \int_{\Omega} \underline{\underline{f}}_v \cdot \delta \underline{\underline{u}} d\Omega$$

Contact term



Two solids in contact

From strong to weak form

- Balance of momentum and boundary conditions

$$\nabla \cdot \underline{\underline{\sigma}} + \underline{\underline{f}}_v = 0 \quad \text{in } \Omega = \Omega_1 \cup \Omega_2 \quad + \text{B.C.}$$

- Balance of virtual works



$$\underbrace{\int_{\Omega} \underline{\underline{\sigma}} \cdot \delta \nabla \underline{\underline{u}} d\Omega}_{\text{Change of the internal energy}} + \underbrace{\int_{\bar{\Gamma}_c^1} \left(\sigma_n \delta g_n + \underline{\underline{\sigma}}_t^T \delta \underline{\underline{\xi}} \right) d\bar{\Gamma}_c^1}_{\text{Contact term}} =$$

Change of the internal energy

Contact term

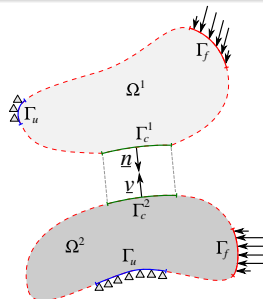
$$\underbrace{\int_{\Gamma_f} \underline{\underline{\sigma}}_0 \cdot \delta \underline{\underline{u}} d\Gamma}_{\text{Virtual work of external forces}} + \underbrace{\int_{\Omega} \underline{\underline{f}}_v \cdot \delta \underline{\underline{u}} d\Omega}_{\text{Virtual work of volume forces}}$$

Virtual work of external forces

Virtual work of volume forces

- Functional space

$\underline{\underline{u}} \in \mathbb{H}^1(\Omega)$ Sobolev space of the first order
and $\underline{\underline{u}}$ satisfy boundary and **contact conditions**.



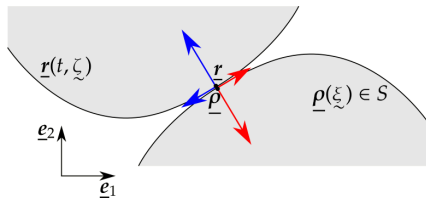
Two solids in contact

Towards variational inequality

■ Contact term

$$\int_{\bar{\Gamma}_c^1} \left(\sigma_n \delta g_n + \underline{\varrho}_t^T \delta \underline{\xi} \right) d\bar{\Gamma}_c^1$$

$$\int_{\bar{\Gamma}_c^1} \sigma_n \delta g_n d\bar{\Gamma}_c^1 \leq 0$$



Contact configuration $\sigma_n \delta g_n = 0$, $\sigma_n \leq 0$

$$\int_{\Omega} \underline{\underline{\sigma}} \cdot \cdot \delta \nabla \underline{u} d\Omega + \int_{\bar{\Gamma}_c^1} \underline{\varrho}_t^T \delta \underline{\xi} d\bar{\Gamma}_c^1 \geq \int_{\Gamma_f} \underline{\sigma}_0 \cdot \delta \underline{u} d\Gamma + \int_{\Omega} \underline{f}_v \cdot \delta \underline{u} d\Omega,$$

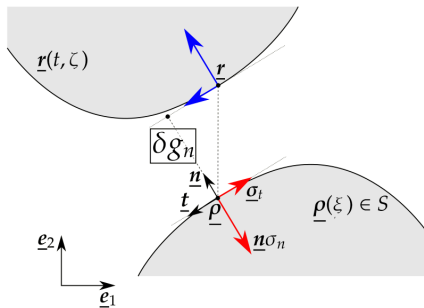
$$\mathbb{K} = \left\{ \delta \underline{u} \in \mathbb{H}^1(\Omega) \mid \delta \underline{u} = 0 \text{ on } \Gamma_u, g_n(\underline{u} + \delta \underline{u}) \geq 0 \text{ on } \Gamma_c \right\}$$

Towards variational inequality

■ Contact term

$$\int_{\bar{\Gamma}_c^1} \left(\sigma_n \delta g_n + \underline{\varrho}_t^T \delta \underline{\xi} \right) d\bar{\Gamma}_c^1$$

$$\int_{\bar{\Gamma}_c^1} \sigma_n \delta g_n d\bar{\Gamma}_c^1 \leq 0$$



Virtual change of the configuration

$$\int_{\Omega} \underline{\underline{\sigma}} \cdot \cdot \delta \nabla \underline{\underline{u}} d\Omega + \int_{\bar{\Gamma}_c^1} \underline{\underline{\varrho}}_t^T \delta \underline{\underline{\xi}} d\bar{\Gamma}_c^1 \geq \int_{\Gamma_f} \underline{\underline{\sigma}}_0 \cdot \delta \underline{\underline{u}} d\Gamma + \int_{\Omega} \underline{\underline{f}}_v \cdot \delta \underline{\underline{u}} d\Omega,$$

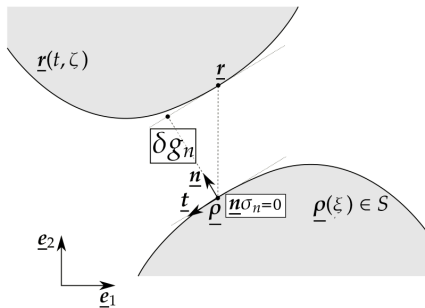
$$\mathbb{K} = \left\{ \delta \underline{\underline{u}} \in \mathbb{H}^1(\Omega) \mid \delta \underline{\underline{u}} = 0 \text{ on } \Gamma_u, g_n(\underline{\underline{u}} + \delta \underline{\underline{u}}) \geq 0 \text{ on } \Gamma_c \right\}$$

Towards variational inequality

■ Contact term

$$\int_{\bar{\Gamma}_c^1} \left(\sigma_n \delta g_n + \underline{\varrho}_t^T \delta \underline{\xi} \right) d\bar{\Gamma}_c^1$$

$$\int_{\bar{\Gamma}_c^1} \sigma_n \delta g_n d\bar{\Gamma}_c^1 \leq 0$$



Normal term in separation $\delta g_n > 0$

$$\int_{\Omega} \underline{\underline{\sigma}} \cdot \delta \nabla \underline{\underline{u}} d\Omega + \int_{\bar{\Gamma}_c^1} \underline{\underline{\varrho}}_t^T \delta \underline{\underline{\xi}} d\bar{\Gamma}_c^1 \geq \int_{\Gamma_f} \underline{\underline{\sigma}}_0 \cdot \delta \underline{\underline{u}} d\Gamma + \int_{\Omega} \underline{\underline{f}}_v \cdot \delta \underline{\underline{u}} d\Omega,$$

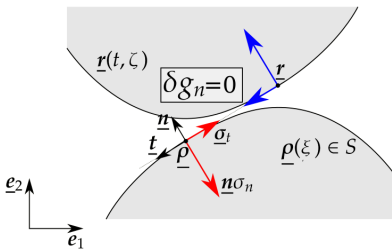
$$\mathbb{K} = \left\{ \delta \underline{\underline{u}} \in \mathbb{H}^1(\Omega) \mid \delta \underline{\underline{u}} = 0 \text{ on } \Gamma_u, g_n(\underline{\underline{u}} + \delta \underline{\underline{u}}) \geq 0 \text{ on } \Gamma_c \right\}$$

Towards variational inequality

■ Contact term

$$\int_{\bar{\Gamma}_c^1} \left(\sigma_n \delta g_n + \underline{\varrho}_t^T \delta \underline{\xi} \right) d\bar{\Gamma}_c^1$$

$$\int_{\bar{\Gamma}_c^1} \sigma_n \delta g_n d\bar{\Gamma}_c^1 \leq 0$$



Normal term in sliding $\delta g_n = 0$

$$\int_{\Omega} \underline{\underline{\sigma}} \cdot \cdot \delta \nabla \underline{\underline{u}} d\Omega + \int_{\bar{\Gamma}_c^1} \underline{\underline{\varrho}}_t^T \delta \underline{\underline{\xi}} d\bar{\Gamma}_c^1 \geq \int_{\Gamma_f} \underline{\underline{\sigma}}_0 \cdot \delta \underline{\underline{u}} d\Gamma + \int_{\Omega} \underline{\underline{f}}_v \cdot \delta \underline{\underline{u}} d\Omega,$$

$$\mathbb{K} = \left\{ \delta \underline{\underline{u}} \in \mathbb{H}^1(\Omega) \mid \delta \underline{\underline{u}} = 0 \text{ on } \Gamma_u, g_n(\underline{\underline{u}} + \delta \underline{\underline{u}}) \geq 0 \text{ on } \Gamma_c \right\}$$

Back to variational equality (unconstrained)

- Constrained minimization problem

$$\int_{\Omega} \underline{\underline{\sigma}} \cdot \delta \nabla \underline{u} d\Omega + \int_{\bar{\Gamma}_c^1} \underline{\underline{\sigma}}_i^T \delta \underline{\underline{\xi}} d\bar{\Gamma}_c^1 \geq \int_{\Gamma_f} \underline{\underline{\sigma}}_0 \cdot \delta \underline{u} d\Gamma + \int_{\Omega} \underline{f}_v \cdot \delta \underline{u} d\Omega,$$

$$\mathbb{K} = \left\{ \delta \underline{u} \in \mathbb{H}^1(\Omega) \mid \delta \underline{u} = 0 \text{ on } \Gamma_u, g_n(\underline{u} + \delta \underline{u}) \geq 0 \text{ on } \Gamma_c \right\}$$

- Use optimization theory to convert to

$$\int_{\Omega} \underline{\underline{\sigma}} \cdot \delta \nabla \underline{u} d\Omega + \int_{\Gamma_c^1} \underbrace{C(\sigma_n, \sigma_t, g_n, \underline{\underline{\xi}}, \delta \underline{u})}_{\text{Contact term}^*} d\Gamma_c^1 = \int_{\Gamma_f} \underline{\underline{\sigma}}_0 \cdot \delta \underline{u} d\Gamma + \int_{\Omega} \underline{f}_v \cdot \delta \underline{u} d\Omega,$$

Unconstrained functional space $\mathbb{K} = \left\{ \delta \underline{u} \in \mathbb{H}^1(\Omega) \mid \delta \underline{u} = 0 \text{ on } \Gamma_u \right\}$

Contact term* is defined on the *potential contact zone* Γ_c^1 .

Optimization methods: recall

Functional to be minimized $F(\mathbf{x})$ under constraint $g(\mathbf{x}) \geq 0$

- Penalty method
- Lagrange multipliers method
- Augmented Lagrangian method

Functional to be minimized $F(\mathbf{x})$ under constraint $g(\mathbf{x}) \geq 0$

■ Penalty method

- New functional

$$F_p(\mathbf{x}) = F(\mathbf{x}) + \boxed{\epsilon \langle -g(\mathbf{x}) \rangle^2} = F(\mathbf{x}) + \begin{cases} 0, & \text{if } g(\mathbf{x}) \geq 0 \quad \text{non-contact} \\ \epsilon g^2(\mathbf{x}), & \text{if } g(\mathbf{x}) < 0 \quad \text{contact} \end{cases}$$

where ϵ is the penalty parameter.

- Stationary point must satisfy

$$\nabla F_p(\mathbf{x}) = \nabla F(\mathbf{x}) + 2\epsilon \langle -g(\mathbf{x}) \rangle \nabla g(\mathbf{x}) = 0$$

- Solution **tends** to the precise solution as $\epsilon \rightarrow \infty$

■ Lagrange multipliers method

■ Augmented Lagrangian method

Macauley brackets $\langle x \rangle = \begin{cases} x, & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases}$

Optimization methods: recall

Functional to be minimized $F(\mathbf{x})$ under constraint $g(\mathbf{x}) \geq 0$

- Penalty method $F_p(\mathbf{x}) = F(\mathbf{x}) + \epsilon \langle -g(\mathbf{x}) \rangle^2$

- **Lagrange multipliers method**

- New functional called **Lagrangian**

$$\mathcal{L}(\mathbf{x}, \lambda) = F(\mathbf{x}) + \lambda g(\mathbf{x})$$

- Saddle point problem

$$\min_{\mathbf{x}} \max_{\lambda} \{\mathcal{L}(\mathbf{x}, \lambda)\} \longrightarrow \mathbf{x}^* \longleftarrow \min_{g(\mathbf{x}) \geq 0} \{F(\mathbf{x})\}$$

- Stationary point

$$\nabla_{\mathbf{x}, \lambda} \mathcal{L} = \begin{bmatrix} \nabla_{\mathbf{x}} F(\mathbf{x}) + \lambda \nabla_{\mathbf{x}} g(\mathbf{x}) \\ g(\mathbf{x}) \end{bmatrix} = 0 \quad \text{need to verify } \lambda \leq 0$$

- **Augmented Lagrangian method**

Macaulay brackets $\langle x \rangle = \begin{cases} x, & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases}$

Optimization methods: recall

Functional to be minimized $F(\mathbf{x})$ under constraint $g(\mathbf{x}) \geq 0$

■ Penalty method $F_p(\mathbf{x}) = F(\mathbf{x}) + \epsilon \langle -g(\mathbf{x}) \rangle^2$

■ Lagrange multipliers method $\mathcal{L}(\mathbf{x}, \lambda) = F(\mathbf{x}) + \lambda g(\mathbf{x})$

■ **Augmented Lagrangian method**

[Hestnes 1969], [Powell 1969], [Glowinski & Le Tallec 1989], [Alart & Curnier 1991], [Simo & Laursen 1992]

• New functional, augmented Lagrangian

$$\mathcal{L}_a(\mathbf{x}, \lambda) = F(\mathbf{x}) + \begin{cases} \lambda g(\mathbf{x}) + \epsilon g^2(\mathbf{x}), & \text{if } \lambda + 2\epsilon g(\mathbf{x}) \geq 0, \text{ contact} \\ -\frac{1}{4\epsilon} \lambda^2, & \text{if } \lambda + 2\epsilon g(\mathbf{x}) < 0, \text{ non-contact} \end{cases}$$

• Stationary point

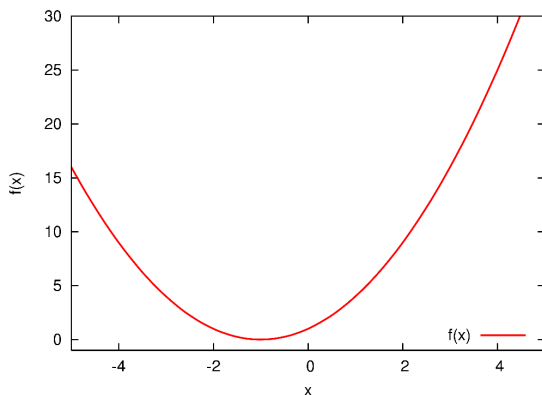
$$\nabla_{\mathbf{x}, \lambda} \mathcal{L}_a = \begin{cases} \begin{bmatrix} \nabla_{\mathbf{x}} F(\mathbf{x}) + \lambda \nabla_{\mathbf{x}} g(\mathbf{x}) + 2\epsilon g(\mathbf{x}) \nabla g(\mathbf{x}) \\ g(\mathbf{x}) \end{bmatrix} = 0, & \text{if contact} \\ \begin{bmatrix} \nabla_{\mathbf{x}} F(\mathbf{x}) \\ -\frac{\lambda}{\epsilon} \end{bmatrix} = 0, & \text{if non-contact} \end{cases}$$

Macaulay brackets $\langle x \rangle = \begin{cases} x, & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases}$



Uzawa algorithm

Optimization methods: example

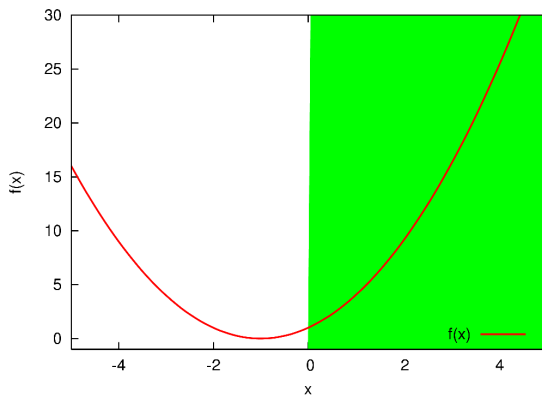


Functional : $f(x) = x^2 + 2x + 1$

Constrain : $g(x) = x \geq 0$

Solution : $x^* = 0$

Optimization methods: example



Functional : $f(x) = x^2 + 2x + 1$

Constrain : $g(x) = x \geq 0$

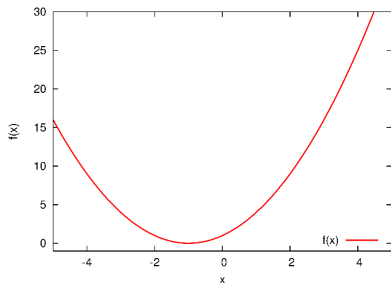
Solution : $x^* = 0$

Penalty method: example

$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

■ Penalty method

$$F_p(x) = F(x) + \epsilon \langle -g(x) \rangle^2$$

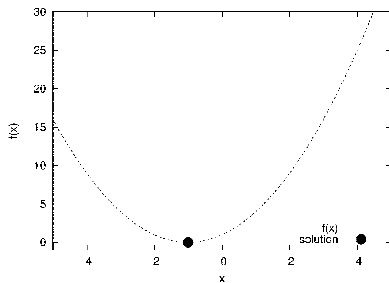
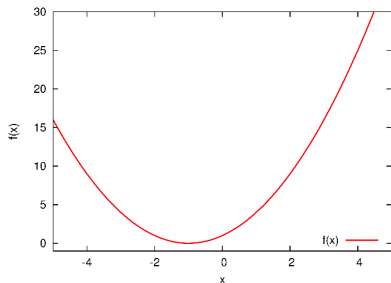


Penalty method: example

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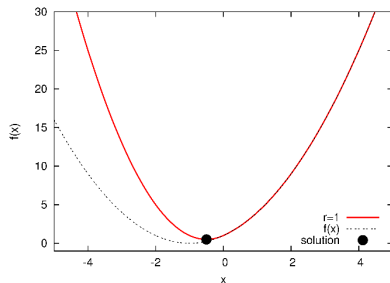
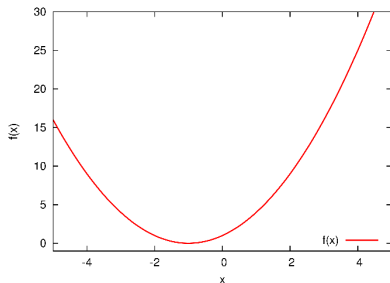
$\epsilon = 0$

Penalty method: example

$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

■ Penalty method

$$F_p(x) = F(x) + \epsilon \langle -g(x) \rangle^2$$



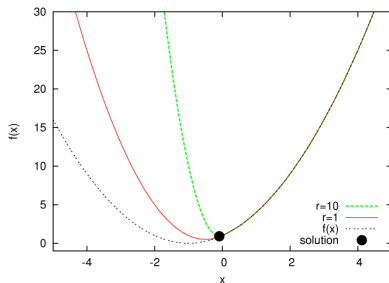
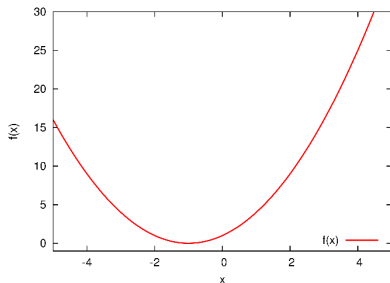
$\epsilon = 1$

Penalty method: example

$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

■ Penalty method

$$F_p(x) = F(x) + \epsilon \langle -g(x) \rangle^2$$



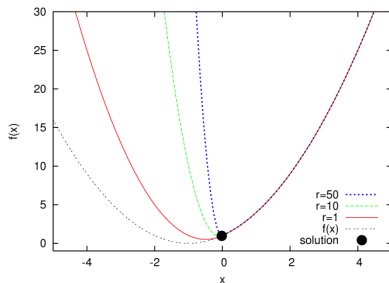
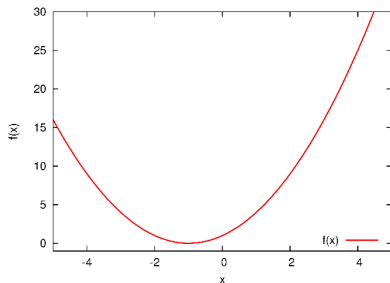
$\epsilon = 10$

Penalty method: example

$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

■ Penalty method

$$F_p(x) = F(x) + \epsilon \langle -g(x) \rangle^2$$



$\epsilon = 50$

Penalty method: example

$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

■ Penalty method

$$F_p(x) = F(x) + \epsilon \langle -g(x) \rangle^2$$

Advantages 😊

- simple physical interpretation
- simple implementation
- no additional degrees of freedom
- “mathematically” smooth functional

Drawbacks 😞

- practically non-smooth functional
- solution is not exact:
 - too small penalty $\rightarrow$ large penetration
 - too large penalty $\rightarrow$ ill-conditioning of the tangent matrix
- user has to choose penalty ϵ properly or automatically and/or adapt during convergence

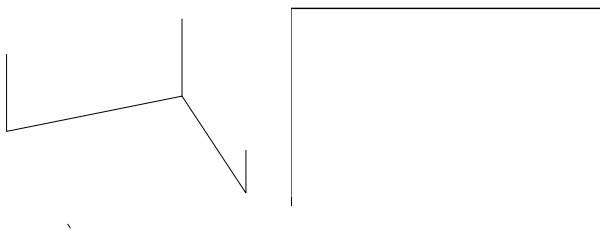
Lagrange multipliers method: example

$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

■ Lagrange multipliers method

$$\mathcal{L}(x, \lambda) = F(x) + \boxed{\lambda g(x)} \rightarrow \text{Saddle point} \rightarrow \min_x \max_{\lambda} \mathcal{L}(x, \lambda)$$

Need to check that $\lambda \leq 0$



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Need to check that $\lambda \leq 0$

Advantages 😊

- exact solution
- no adjustable parameters

Drawbacks ☹

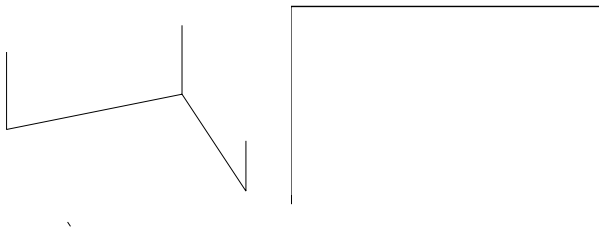
- Lagrangian is not smooth
- additional degrees of freedom
- not fully unconstrained: $\lambda \leq 0$

Augmented Lagrangian method: example

$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

■ Augmented Lagrangian method

$$\mathcal{L}_a(\mathbf{x}, \lambda) = F(\mathbf{x}) + \begin{cases} \lambda g(\mathbf{x}) + \epsilon g^2(\mathbf{x}), & \text{if } \lambda + 2\epsilon g(\mathbf{x}) \geq 0, \text{ contact} \\ -\frac{1}{4\epsilon} \lambda^2, & \text{if } \lambda + 2\epsilon g(\mathbf{x}) < 0, \text{ non-contact} \end{cases}$$



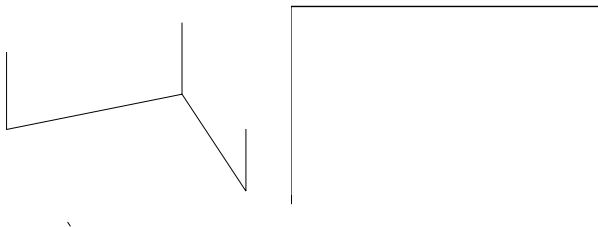
Yellow line separates contact and non-contact regions

Augmented Lagrangian method: example

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■ Augmented Lagrangian method

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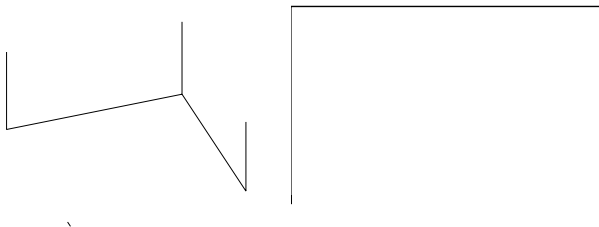
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Advantages 😊

- exact solution
- smooth functional (!)
- fully unconstrained

Drawbacks 😞

- additional degrees of freedom
- quite sensitive to parameter ϵ
- need to adjust ϵ during convergence

Application to contact problems: weak form

$$\int_{\Omega} \underline{\underline{\sigma}} \cdot \cdot \delta \nabla \underline{\underline{u}} d\Omega + \underbrace{\int_{\Gamma_c^1} \boxed{C}}_{\text{Contact term}} d\Gamma_c^1 = \int_{\Gamma_f} \underline{\underline{\sigma}}_0 \cdot \delta \underline{\underline{u}} d\Gamma + \int_{\Omega} \underline{\underline{f}}_v \cdot \delta \underline{\underline{u}} d\Omega,$$

$$\mathbb{K} = \left\{ \delta \underline{\underline{u}} \in \mathbb{H}^1(\Omega) \mid \delta \underline{\underline{u}} = 0 \text{ on } \Gamma_u \right\}$$

■ Penalty method

$$\text{Pressure: } \sigma_n = \epsilon g_n, \text{ Shear: } \underline{\underline{\sigma}}_t = \begin{cases} \epsilon \underline{\underline{g}}_t, & \text{if stick } |\sigma_t| < \mu |\sigma_n| \\ \mu \epsilon g_n \delta \underline{\underline{g}}_t / |\delta \underline{\underline{g}}_t|, & \text{if slip } |\sigma_t| = \mu |\sigma_n| \end{cases}$$

Contact term

$$C = C(g_n, \underline{\underline{g}}_t, \delta g_n, \delta \underline{\underline{g}}_t) = \sigma_n \delta g_n + \underline{\underline{\sigma}}_t \cdot \delta \underline{\underline{g}}_t$$

Application to contact problems: weak form

$$\int_{\Omega} \underline{\underline{\sigma}} \cdot \cdot \delta \nabla \underline{u} d\Omega + \underbrace{\int_{\Gamma_c^1} \boxed{C}}_{\text{Contact term}} d\Gamma_c^1 = \int_{\Gamma_f} \underline{\sigma}_0 \cdot \delta \underline{u} d\Gamma + \int_{\Omega} \underline{f}_v \cdot \delta \underline{u} d\Omega,$$

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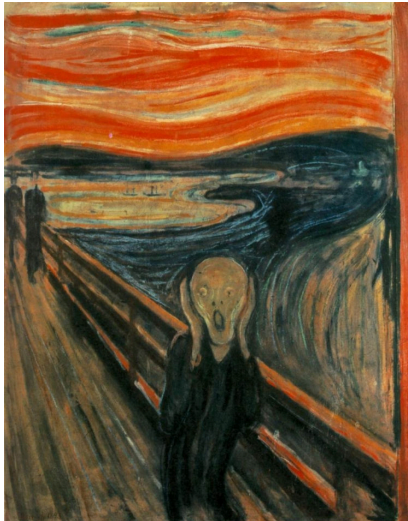
■ Augmented Lagrangian method

Contact term

$$C = C(g_n, \underline{g}_t, \lambda_n, \underline{\lambda}_t, \delta g_n, \delta \underline{g}_t, \delta \lambda_n, \delta \underline{\lambda}_t)$$

$$C = \begin{cases} -\frac{1}{\epsilon} \left(\lambda_n \delta \lambda_n - \underline{\lambda}_t \cdot \delta \underline{\lambda}_t \right), & \text{if non-contact } \lambda_n + \epsilon g_n \geq 0 \\ \hat{\lambda}_n \delta g_n + g_n \delta \lambda_n + \underline{\hat{\lambda}}_t \cdot \delta \underline{g}_t + \underline{g}_t \cdot \delta \underline{\hat{\lambda}}_t, & \text{if stick } |\underline{\hat{\lambda}}_t| \leq \mu |\hat{\sigma}_n| \\ \hat{\lambda}_n \delta g_n + g_n \delta \lambda_n + \mu \hat{\sigma}_n - \mu \hat{\sigma}_n \frac{\hat{\lambda}_t}{|\underline{\hat{\lambda}}_t|} \cdot \delta \underline{g}_t - \frac{1}{\epsilon} \left(\lambda_t + \mu \hat{\sigma}_n \frac{\hat{\lambda}_t}{|\underline{\hat{\lambda}}_t|} \right) \cdot \delta \underline{\lambda}_t, & \text{if slip } |\underline{\hat{\lambda}}_t| \geq \mu |\hat{\sigma}_n| \end{cases}$$

where $\hat{\lambda}_n = \lambda_n + \epsilon g_n$ and $\underline{\hat{\lambda}}_t = \underline{\lambda}_t + \epsilon \underline{g}_t$.



The scream

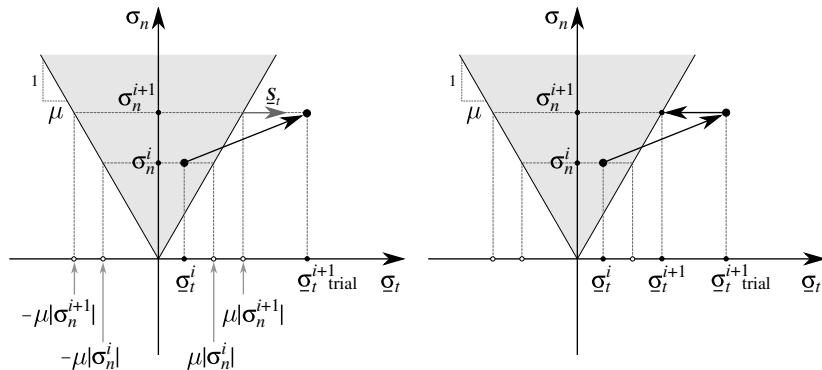
Friction: methods

- Optimization methods: penalty or augmented Lagrangian method
- Return mapping algorithm for penalty
- Analogy with elasto-plastic formulation problem^[1]

[1] Curnier "A theory of friction" Int J Solids Struct 20 (1984)

Friction: Return mapping algorithm

■ Return mapping algorithm in 2D

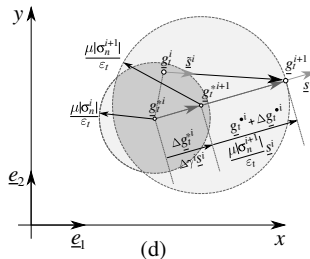
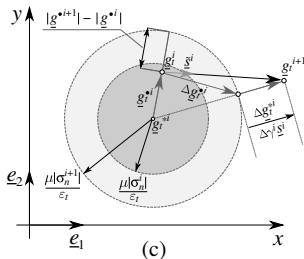
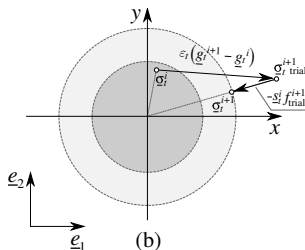
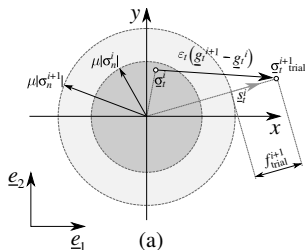


Analogy with non-associated plastic flow^[2]

[2] Curnier A. A theory of friction. International Journal of Solids and Structures 20 (1984)

Friction: Return mapping algorithm

Return mapping algorithm in 3D



A slightly more messy thing

Application to contact problems: linearization

- Non-linear equation

$$R(\underline{u}, \underline{f}) = 0$$

- Contains $\delta g_n, \delta \underline{g}_t$
- Use Newton-Raphson method
- Initial state at step i

$$R(\underline{u}^i, \underline{f}^i) = 0$$

- Should be also satisfied at step $i + 1$

$$R(\underline{u}^{i+1}, \underline{f}^{i+1}) = R(\underline{u}^i + \delta \underline{u}, \underline{f}^{i+1}) = 0$$

- Linearize

$$R(\underline{u}^i + \delta \underline{u}, \underline{f}^{i+1}) = R(\underline{u}^i, \underline{f}^{i+1}) + \frac{\partial R(\underline{u})}{\partial \underline{u}} \delta \underline{u} = 0$$

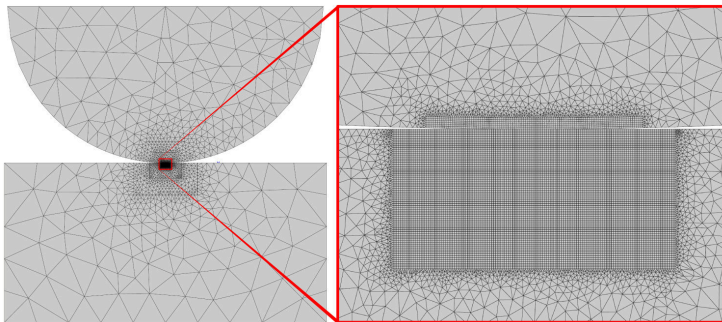
- Finally

$$\delta \underline{u} = - \underbrace{\left[\frac{\partial R(\underline{u})}{\partial \underline{u}} \right]^{-1}}_{\text{contains } \Delta \delta g_n, \Delta \delta \underline{g}_t} R(\underline{u}^i)$$

- NB: Contact problem does not satisfy conditions of Kantorovich theorem on the convergence of Newton's method.

Particularities: mesh and convergence

- Strong **mesh refinement** is required
 - especially at **unknown edges** of contact zones

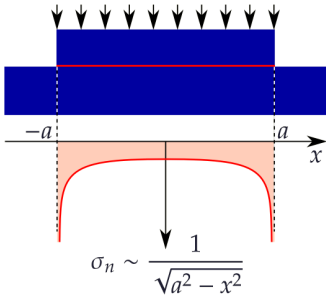


Typical mesh for fretting analysis [L. Sun, H. Proudhon, G. Cailletaud, 2011]

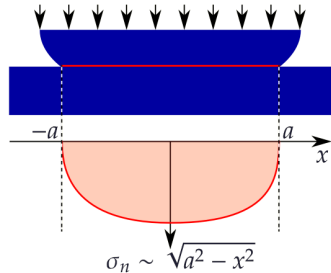
2D ~ 30 000 DoFs, 3D ~ 5 000 000 DoFs

Particularities: mesh and convergence

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$$\sigma_n \xrightarrow{x \rightarrow a} -\infty \quad \left| \frac{\partial \sigma_n}{\partial x} \right| \xrightarrow{x \rightarrow a} \infty$$



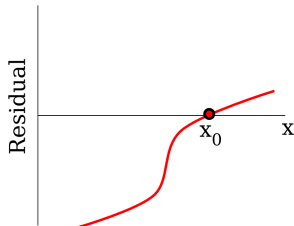
$$\left| \frac{\partial \sigma_n}{\partial x} \right| \xrightarrow{x \rightarrow a} \infty$$

Infinite contact pressure and/or its derivative

Particularities: mesh and convergence

- Strong **mesh refinement** is required
 - especially at **unknown edges** of contact zones
- **Slow change** of boundary conditions:
 - strong non-linearities of contact / friction problems
 - non-uniqueness of solution for frictional problems

Infinite looping

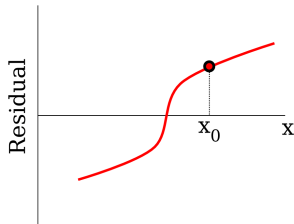


Initial guess $R(x_0, f_0) = 0$

Particularities: mesh and convergence

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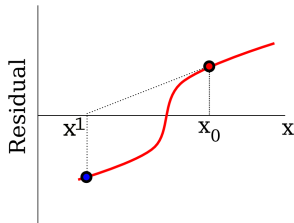


Too rapid change in boundary conditions $R(x_0, f_1) \neq 0$

Particularities: mesh and convergence

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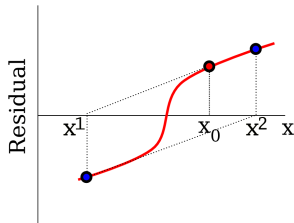
Iterations of Newton-Raphson method

$$R(x_0, f_1) + \left. \frac{\partial R}{\partial x} \right|_{x_0} \delta x = 0 \rightarrow \delta x = - \left. \frac{\partial R}{\partial x} \right|_{x_0}^{-1} R(x_0, f_1) \rightarrow x^1 = x_0 + \delta x$$

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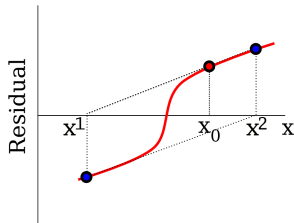
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Particularities: mesh and convergence

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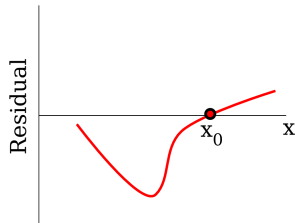


Infinite looping

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Convergence to a “false” solution

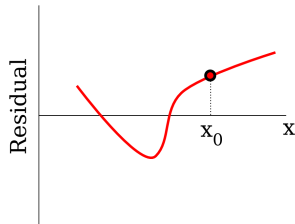


Initial guess $R(x_0, f_0) = 0$

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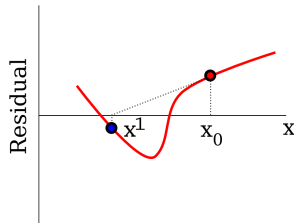


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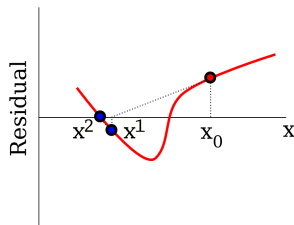
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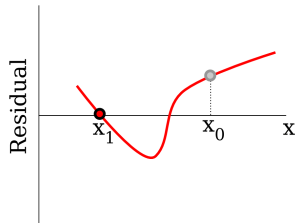
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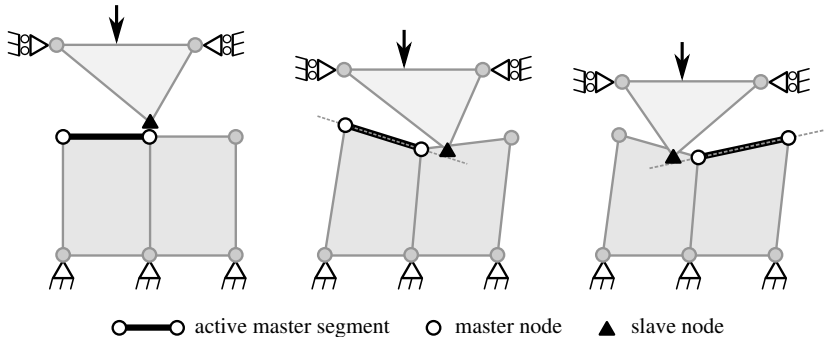
Convergence to a “false” solution



Convergence, but is it a “true” solution ?

Convergence problems: examples

■ Infinite looping, e.g.

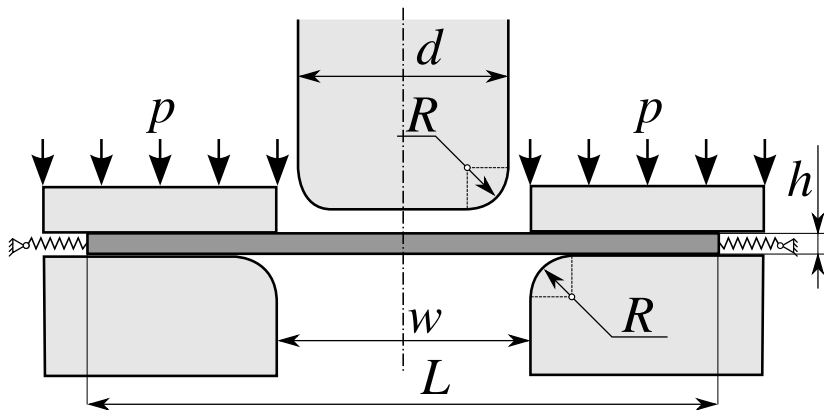


- Change of the contact state (contact/non-contact, stick/slip)
- Interplay between stiffness, friction and augmented Lagrangian coefficients^[1]
- Combination of non-linearities (e.g., plasticity+contact)

Alart P, Méthode de Newton généralisée en mécanique du contact Journal de Mathématiques Pures et Appliquées 76 (1997)

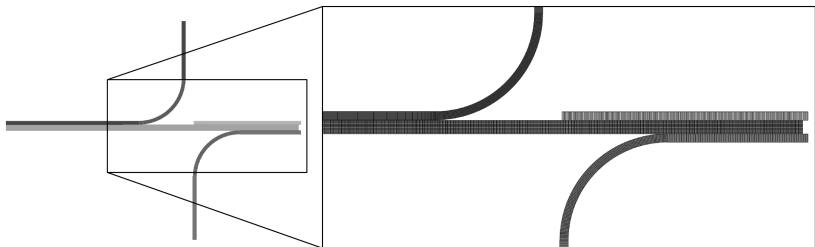
Convergence problems: examples

- Simulation of a deep drawing problem
- Dinite strain plasticity + frictional contact



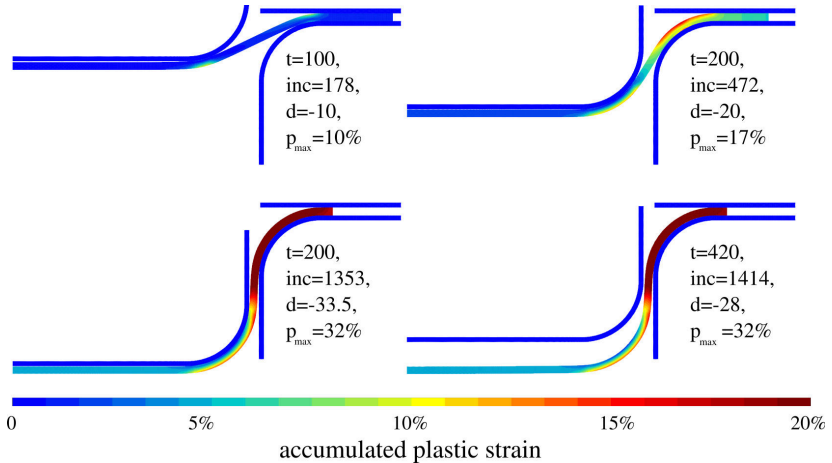
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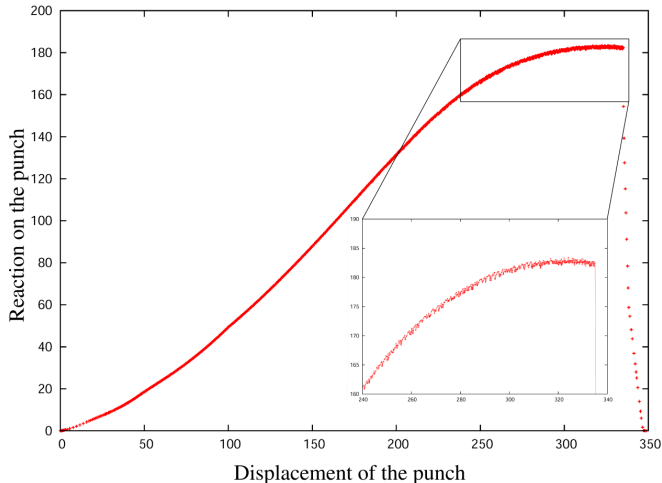
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- Simulation of a deep drawing problem
- Dinite strain plasticity + frictional contact



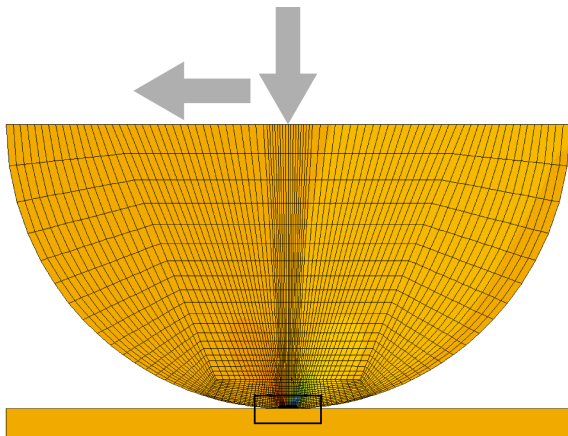
Convergence problems: examples

- Simulation of a deep drawing problem
- Finite strain plasticity + frictional contact



Cylinder-plane frictional contact

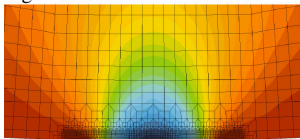
- Non-conservative problem, history of loading is crucial



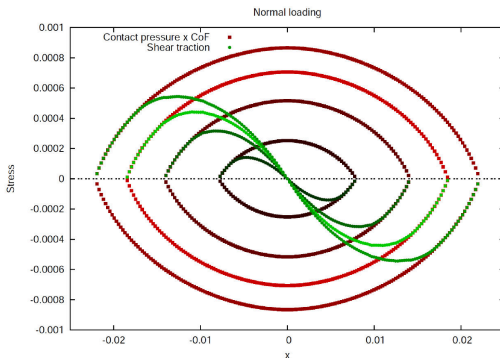
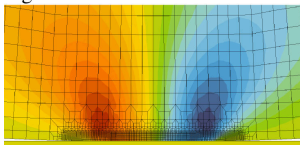
Cylinder-plane frictional contact

- Non-conservative problem, history of loading is crucial

sig22 at maximal normal load



sig12 at maximal normal load

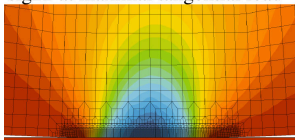


Press in 100 increments, $u_z \sim t^2$

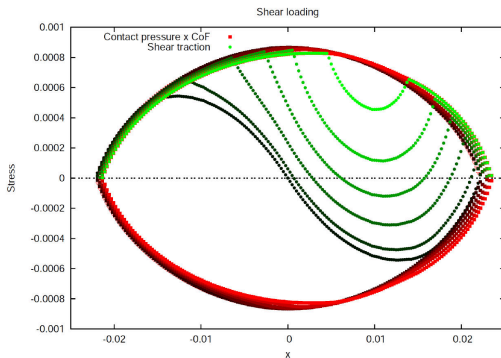
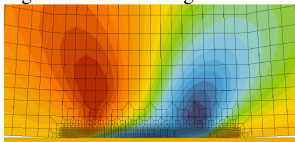
Cylinder-plane frictional contact

- Non-conservative problem, history of loading is crucial

sig22 at maximal tangential load



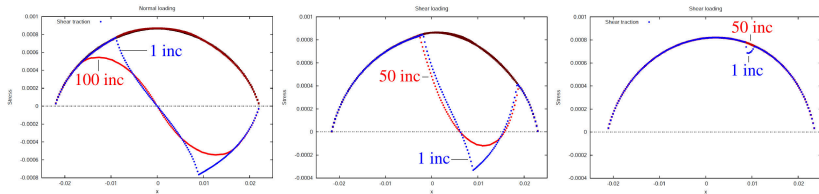
sig12 at maximal tangential load



Shift in 100 increments, $u_z \sim t$

Cylinder-plane frictional contact

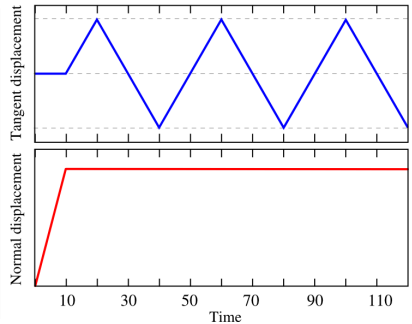
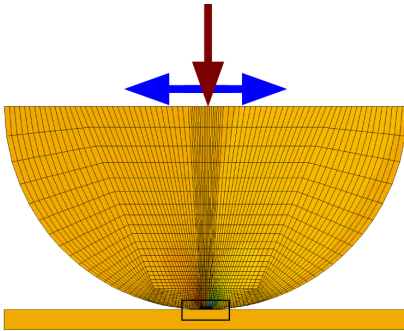
- Non-conservative problem, history of loading is crucial



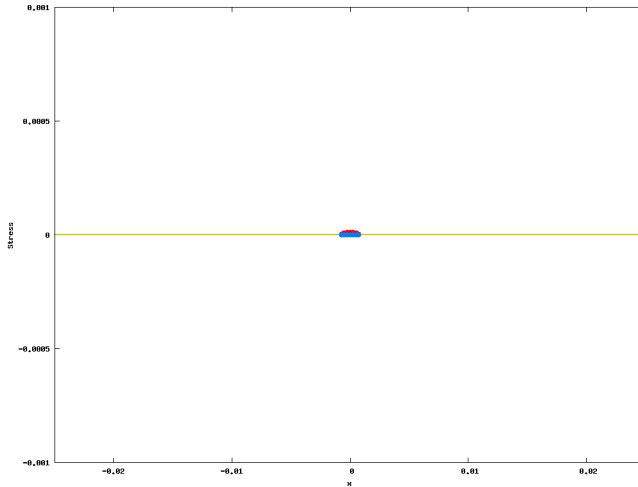
Comparison with: press in 1 increment, shift in 2 increments

Before stick every point of the contact interface has to pass through the slip zone. It is impossible when loaded too fast.

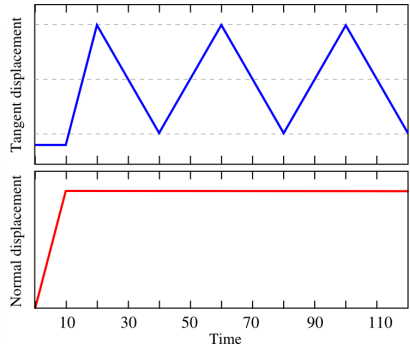
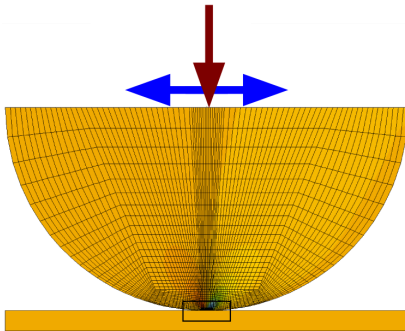
Sphere-plane frictional contact: cycling



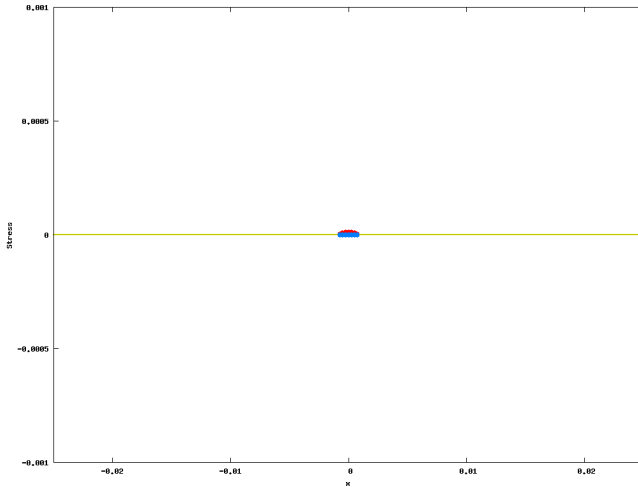
Sphere-plane frictional contact: cycling



Sphere-plane frictional contact: cycling

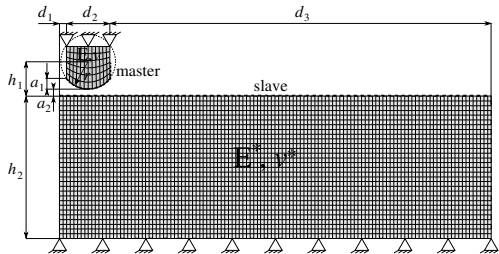


Sphere-plane frictional contact: cycling



Shallow ironing test

- Deformable-on-deformable frictional sliding
- Results obtained by different groups^{1,2,3,4,5,6} differ significantly
- Local and global friction coefficients may differ



[1] Fischer K. A., Wriggers P., "Mortar based frictional contact formulation for higher order interpolations using the moving friction cone", Computer Methods in Applied Mechanics and Engineering, vol. 195, p. 5020-5036, 2006.

[2] Hartmann S., Oliver J., Cante J. C., Weyler R., Hernández J. A., "A contact domain method for large deformation frictional contact problems. Part 2: Numerical aspects", Computer Methods in Applied Mechanics and Engineering, vol. 198, p. 2607-2631, 2009.

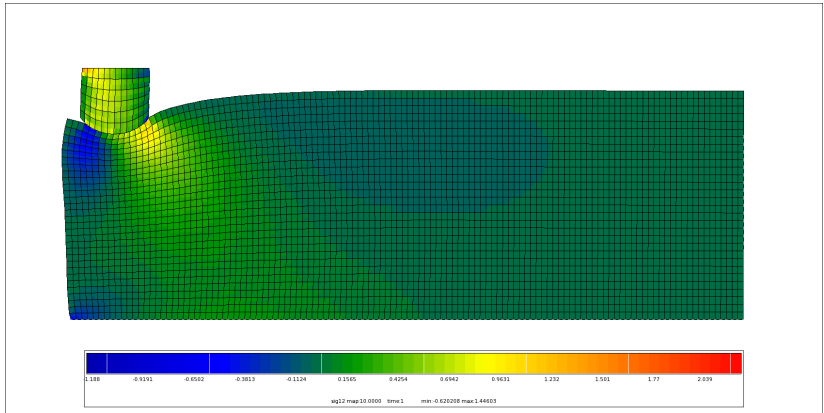
[3] Yastrebov V. A., "Computational contact mechanics: geometry, detection and numerical techniques", Thèse CdM & Onera, 2011.

[4] Kudawoo A. D., "Problèmes industriels de grande dimension en mécanique numérique du contact : performance, fiabilité et robustesse", Thèse @ LMA & LAMSID, 2012.

[5] Poullos K., Renard Y., "A non-symmetric integral approximation of large sliding frictional contact problems of deformable bodies based on ray-tracing", soumis, 2014.

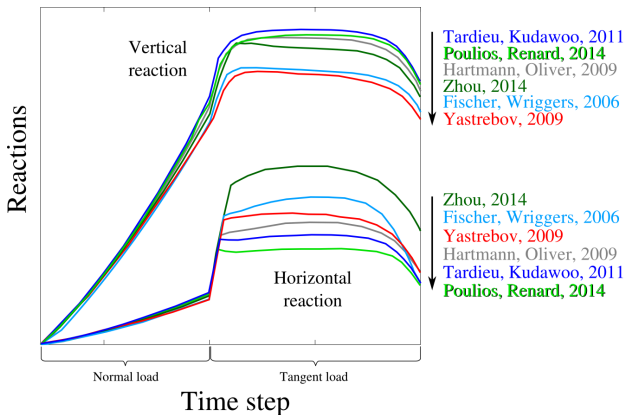
[6] Zhou Lei's blog, <http://kt2008plus.blogspot.de>

Shallow ironing test



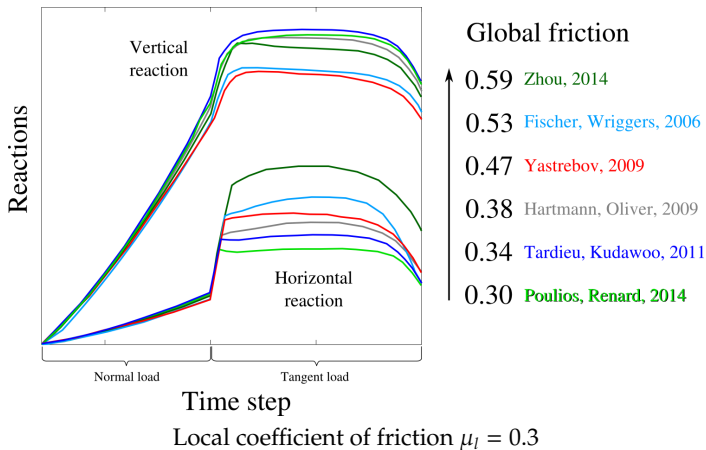
Shallow ironing test

- No agreement between authors
- Dif. authors used dif. meshes (quadrilateral lin./sq., triangular lin.)
- Dif. authors used either finite or infinitesimal strain formulation



Shallow ironing test

- No agreement between authors
- Dif. authors used dif. meshes (quadrilateral lin./sq., triangular lin.)
- Dif. authors used either finite or infinitesimal strain formulation



Examples of contact problems

With analytical solution

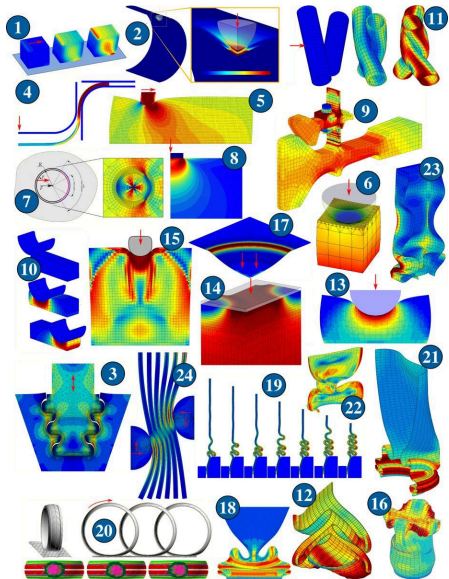
- ★ linear elasticity
- ★ with/without friction

From literature

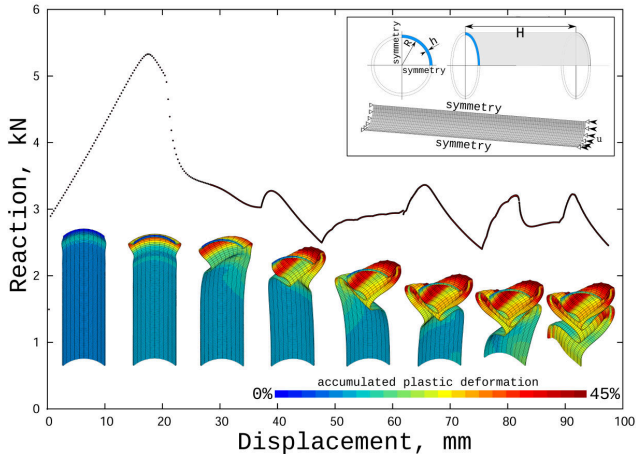
- ★ post-buckling 2D
- ★ finite strains
- ★ elasticity / plasticity
- ★ with/without friction

New

- ★ multi-contacts
- ★ post-buckling 3D
- ★ finite strains
- ★ elasticity / plasticity
- ★ with/without friction



Self-contact problem



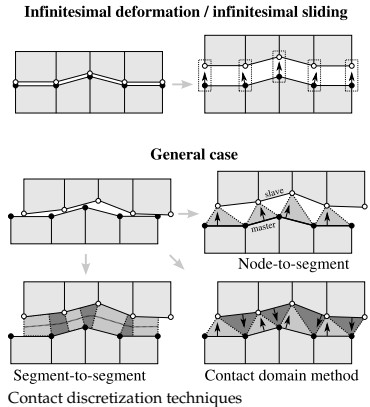
Finite element analysis of a post-buckling behavior of a thin walled tube

Collection of non-linearities: buckling instability, self-contact, finite strain plasticity



- It's just a tip of the “Computational Contact Mechanics” iceberg
- Contact detection
- Contact discretization and integration
- Smoothing techniques
- Energy conservative methods for dynamics

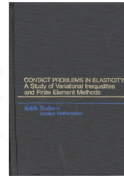
See Basava R. Akula presentation on Friday



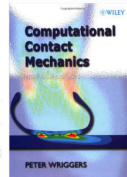
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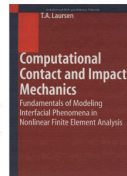
Kikuchi, Oden (1988)



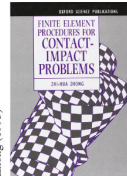
Wriggers (2002)



Laursen (2002)



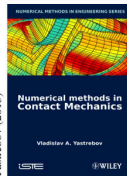
Zhong (1993)



Wriggers, 2nd ed. (2006)



Yastrebov (2013)



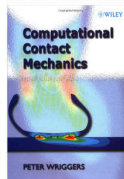
- It's just a tip of the “Computational Contact Mechanics” iceberg
- Contact detection
- Contact discretization and integration

See Basava R. Akula presentation on Friday
- Smoothing techniques
- Energy conservative methods for dynamics
- Several advanced topics

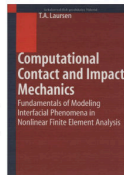
Kikuchi, Oden (1988)



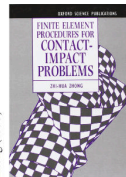
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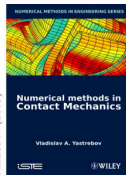
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Wriggers, 2nd ed. (2006)



Yastrebov (2013)



$\mathcal{L}_a(x, \lambda)$

Thank you for your attention!
