Contact mechanics and elements of tribology

Lecture 6.
Computational Contact Mechanics

Vladislav A. Yastrebov

MINES ParisTech, PSL University, CNRS
Centre des Matériaux, Evry, France

@ Centre des Matériaux (& virtually)
February 9, 2022



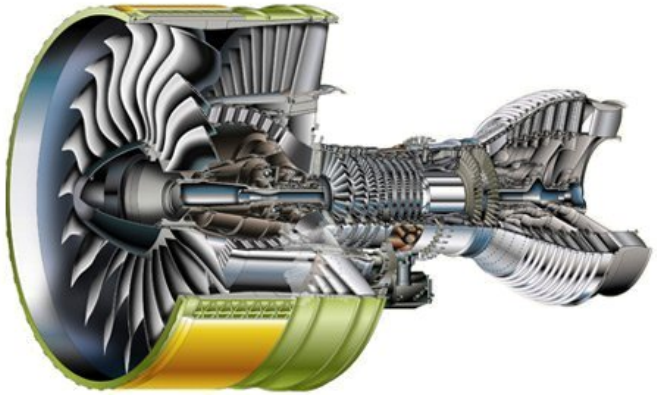
Creative Commons BY
Vladislav A. Yastrebov

Outline

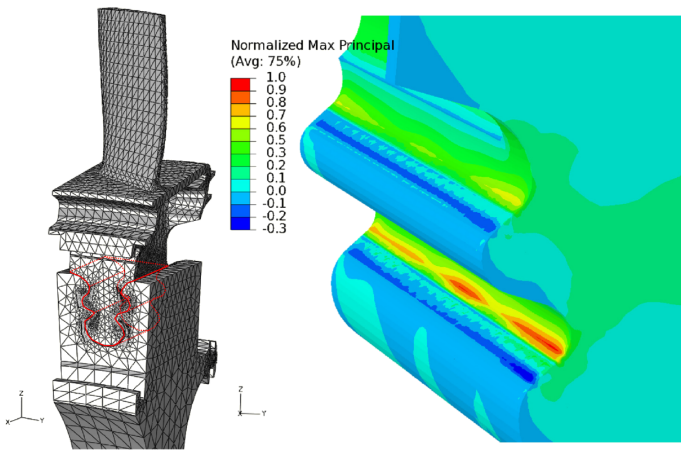
- Introduction
- Governing equations
- Optimization methods
- Resolution algorithm
- Examples

Industrial and natural contact problems

1. Assembled parts, e.g. engines



Aircraft's engine GP 7200
www.safran-group.com



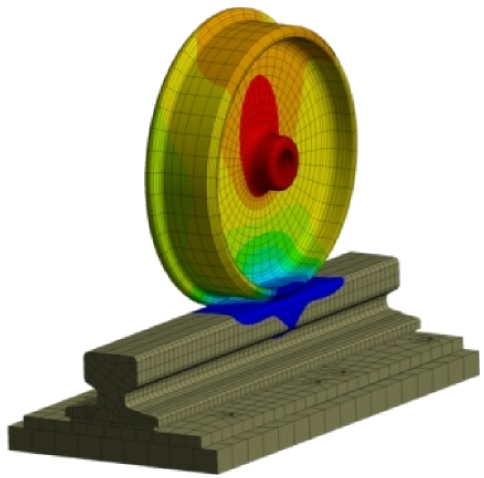
[1] M. W. R. Savage
J. Eng. Gas Turb. Power, 134:012501 (2012)

Industrial and natural contact problems

1. Assembled parts, e.g. engines
2. Railroad contacts



High speed train TGV `www.sncf.com`



Wilde/ANSYS `wildeanalysis.co.uk`

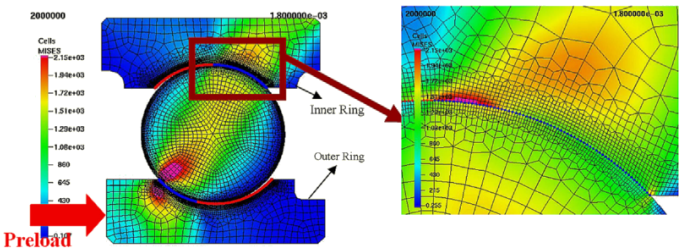
Industrial and natural contact problems

1. Assembled parts, e.g. engines
2. Railroad contacts
3. Gears and bearings



Bearings

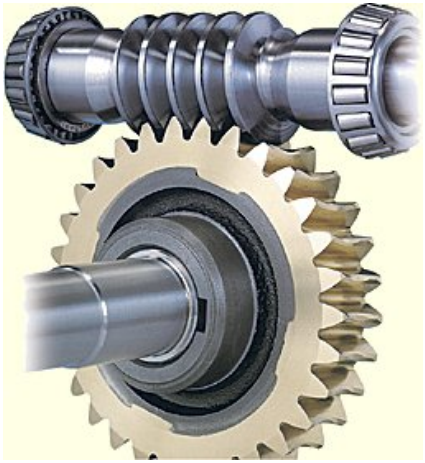
www.skf.com



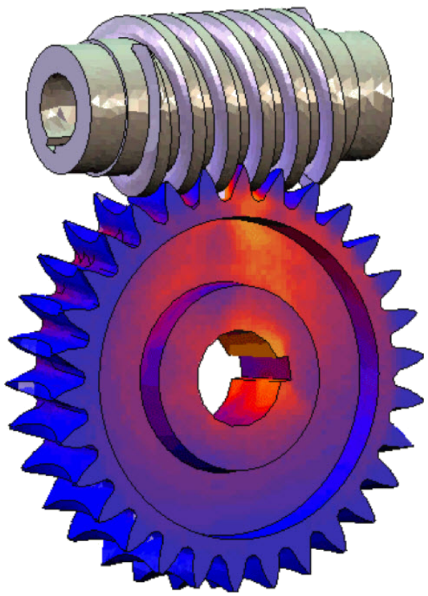
[1] F. Massi, J. Rocchi, A. Culla, Y. Berthier
Mech. Syst. Signal Pr., 24:1068-1080 (2010)

Industrial and natural contact problems

1. Assembled parts, e.g. engines
2. Railroad contacts
3. Gears and bearings



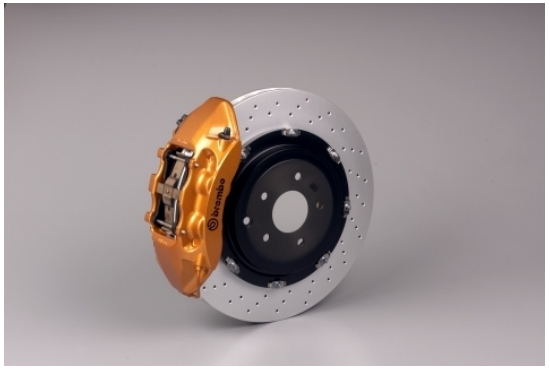
Helical gear www.tpg.com.tw



www.mscsoftware.com

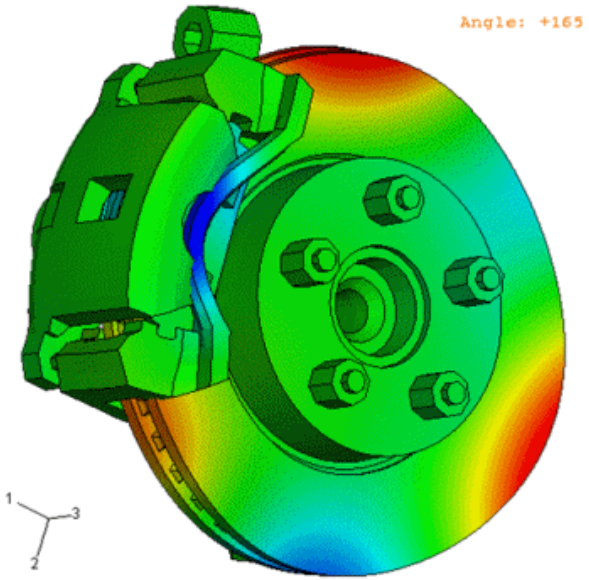
Industrial and natural contact problems

1. Assembled parts, e.g. engines
2. Railroad contacts
3. Gears and bearings
4. Breaking systems



Assembled breaking system

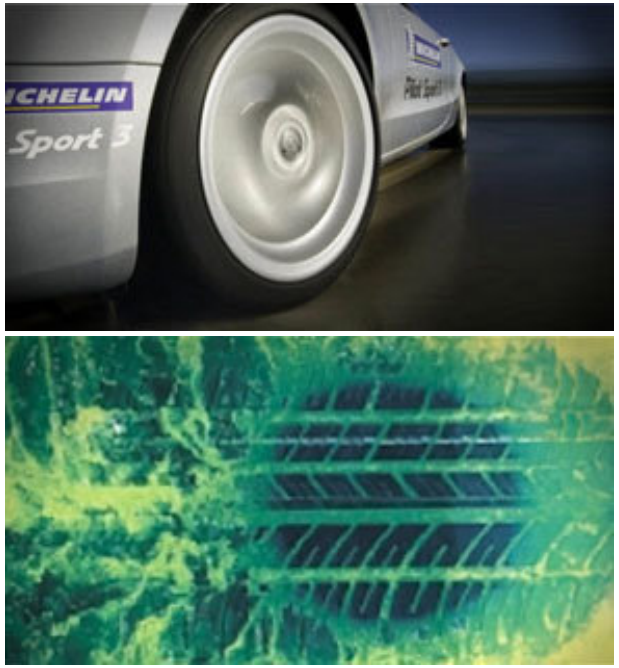
www.brembo.com



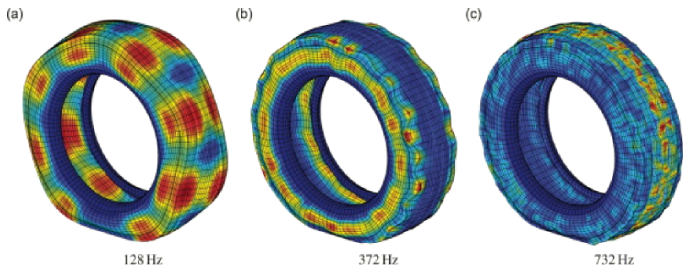
www.mechanicalengineeringblog.com

Industrial and natural contact problems

1. Assembled parts, e.g. engines
2. Railroad contacts
3. Gears and bearings
4. Breaking systems
5. Tire-road contact



Tire-road contact www.michelin.com



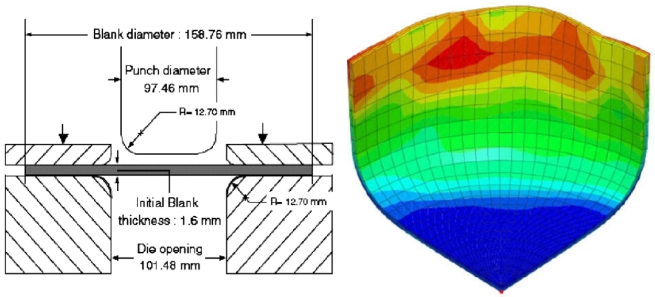
[1] M. Brinkmeier, U. Nackenhorst, S. Petersen, O. von Estorff, J. Sound Vib., 309:20-39 (2008)

Industrial and natural contact problems

1. Assembled parts, e.g. engines
2. Railroad contacts
3. Gears and bearings
4. Breaking systems
5. Tire-road contact
6. Metal forming



Deep drawing www.thomasnet.com



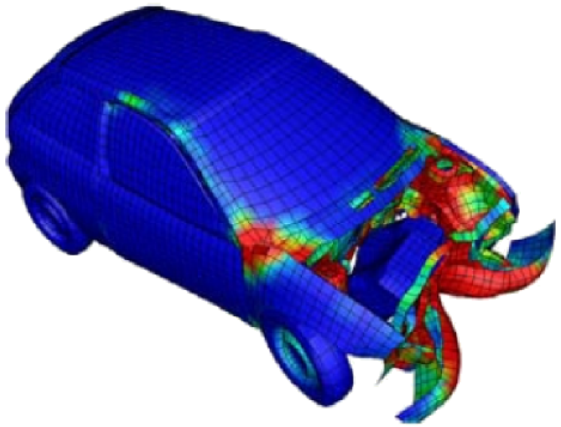
[1] G. Rousselier, F. Barlat, J. W. Yoon
Int. J. Plasticity, 25:2383-2409 (2009)

Industrial and natural contact problems

1. Assembled parts, e.g. engines
2. Railroad contacts
3. Gears and bearings
4. Breaking systems
5. Tire-road contact
6. Metal forming
7. Crash tests



Crash-test www.porsche.com



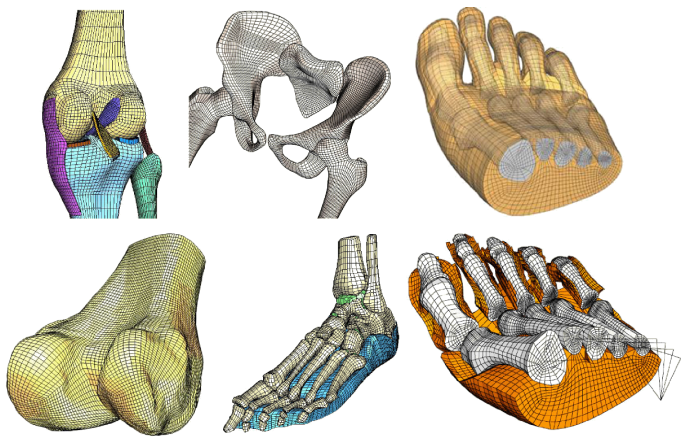
[1] O. Klyavin, A. Michailov, A. Borovkov
www.fea.ru

Industrial and natural contact problems

1. Assembled parts, e.g. engines
2. Railroad contacts
3. Gears and bearings
4. Breaking systems
5. Tire-road contact
6. Metal forming
7. Crash tests
8. Biomechanics



Human articulations
www.sportssupplements.net



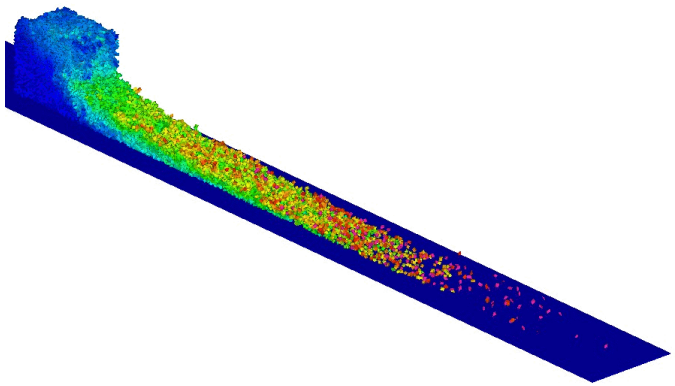
J. A. Weiss, University of Utah
Musculoskeletal Research Laboratories

Industrial and natural contact problems

1. Assembled parts, e.g. engines
2. Railroad contacts
3. Gears and bearings
4. Breaking systems
5. Tire-road contact
6. Metal forming
7. Crash tests
8. Biomechanics
9. Granular materials



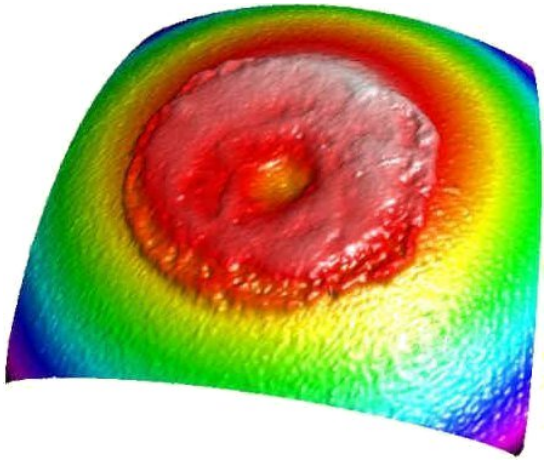
Sand dunes www.en.wikipedia.org



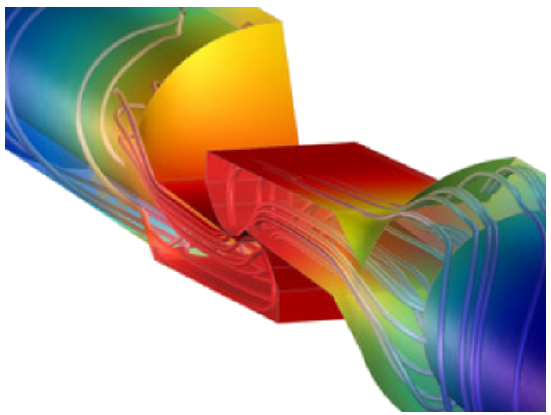
E. Azema et al, LMGC90

Industrial and natural contact problems

1. Assembled parts, e.g. engines
2. Railroad contacts
3. Gears and bearings
4. Breaking systems
5. Tire-road contact
6. Metal forming
7. Crash tests
8. Biomechanics
9. Granular materials
10. Electric contacts



Damage at electric contact zone
www.taicaan.com



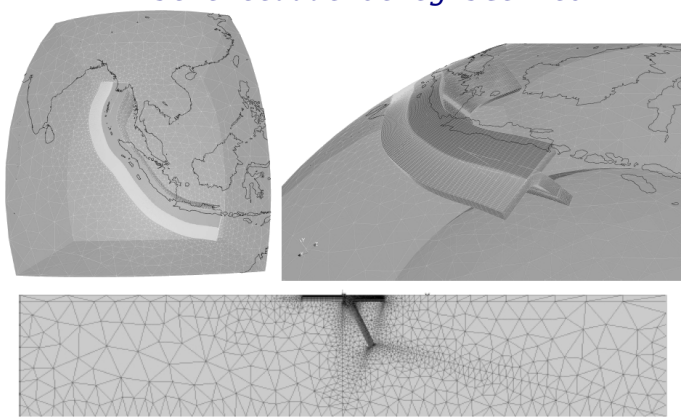
Simulation of electric current
www.comsol.com

Industrial and natural contact problems

1. Assembled parts, e.g. engines
2. Railroad contacts
3. Gears and bearings
4. Breaking systems
5. Tire-road contact
6. Metal forming
7. Crash tests
8. Biomechanics
9. Granular materials
10. Electric contacts
11. Tectonic motions



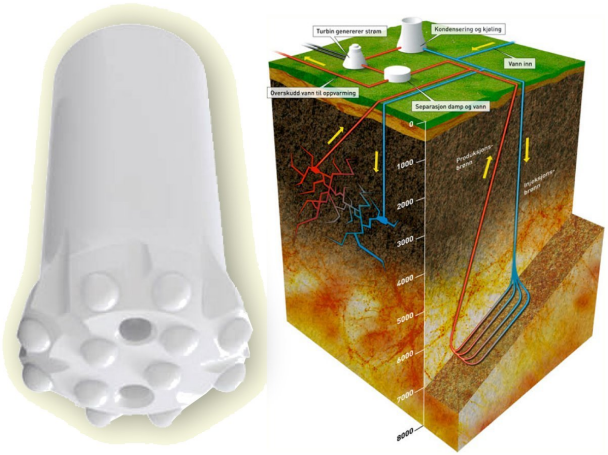
San-Andreas fault, by M. Rightmire
www.sciencedude.ocregister.com



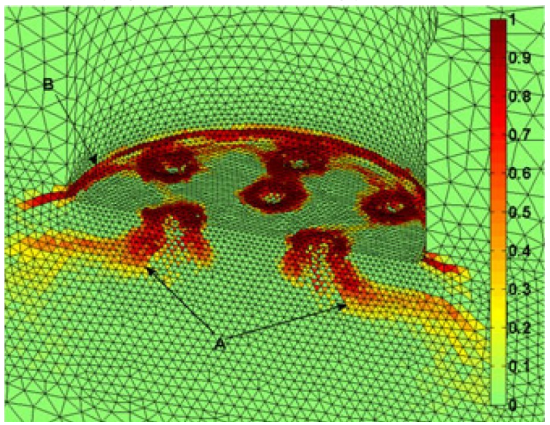
[1] J.D. Garaud, L. Fleitout, G. Cailletaud
Colloque CSMA (2009)

Industrial and natural contact problems

1. Assembled parts, e.g. engines
2. Railroad contacts
3. Gears and bearings
4. Breaking systems
5. Tire-road contact
6. Metal forming
7. Crash tests
8. Biomechanics
9. Granular materials
10. Electric contacts
11. Tectonic motions
12. Deep drilling



Drill Bit tool `RobitRocktools`;
extraction of geothermal energy (`SINTEF, NTNU`*)*



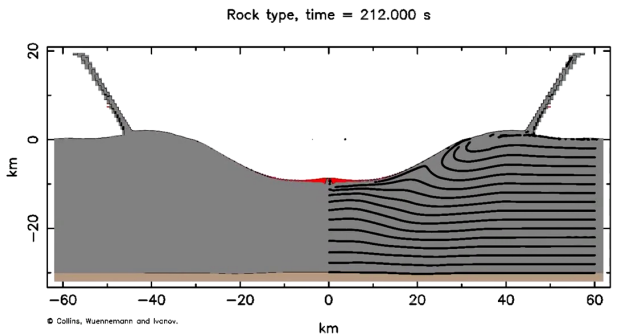
[1] T. Saksala, Int. J. Numer. Anal. Meth. Geomech. (2012)

Industrial and natural contact problems

1. Assembled parts, e.g. engines
2. Railroad contacts
3. Gears and bearings
4. Breaking systems
5. Tire-road contact
6. Metal forming
7. Crash tests
8. Biomechanics
9. Granular materials
10. Electric contacts
11. Tectonic motions
12. Deep drilling
13. Impact and fragmentation



Impact crater, Arizona
www.MrEclipse.com et maps.google.com



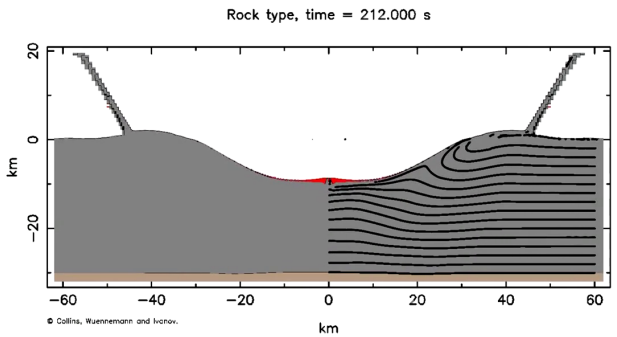
Simulation of formation of Copernicus crater
Yue Z., Johnson B. C., et al. Projectile remnants in central peaks of lunar impact craters. Nature Geo 6 (2013)

Industrial and natural contact problems

1. Assembled parts, e.g. engines
2. Railroad contacts
3. Gears and bearings
4. Breaking systems
5. Tire-road contact
6. Metal forming
7. Crash tests
8. Biomechanics
9. Granular materials
10. Electric contacts
11. Tectonic motions
12. Deep drilling
13. Impact and fragmentation
14. etc.



Impact crater, Arizona
www.MrEclipse.com et maps.google.com



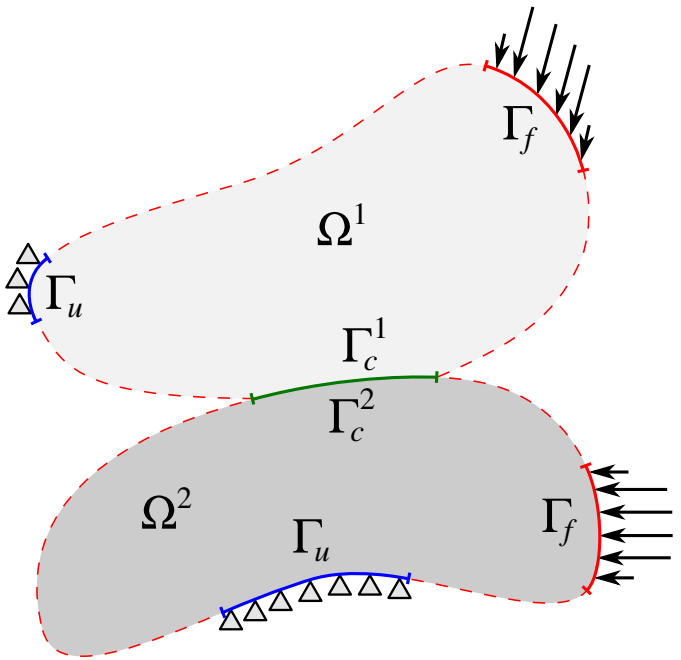
Simulation of formation of Copernicus crater
Yue Z., Johnson B. C., et al. Projectile remnants in central peaks of lunar impact craters. Nature Geo 6 (2013)

Equilibrium and contact conditions

- **Balance of momentum**

$$\begin{cases} \nabla \cdot \underline{\underline{\sigma}} + \underline{f}_v = 0 & \text{in } \Omega_{1,2} \\ \underline{\underline{\sigma}} \cdot \underline{n} = \underline{t}_0 & \text{on } \Gamma_f \\ \underline{u} = \underline{u}_0 & \text{on } \Gamma_u \\ \quad ? & \text{on } \Gamma_c \end{cases}$$

- **Frictionless contact conditions *(intuitive)***
  1. No penetration
  2. No adhesion
  3. No shear transfer

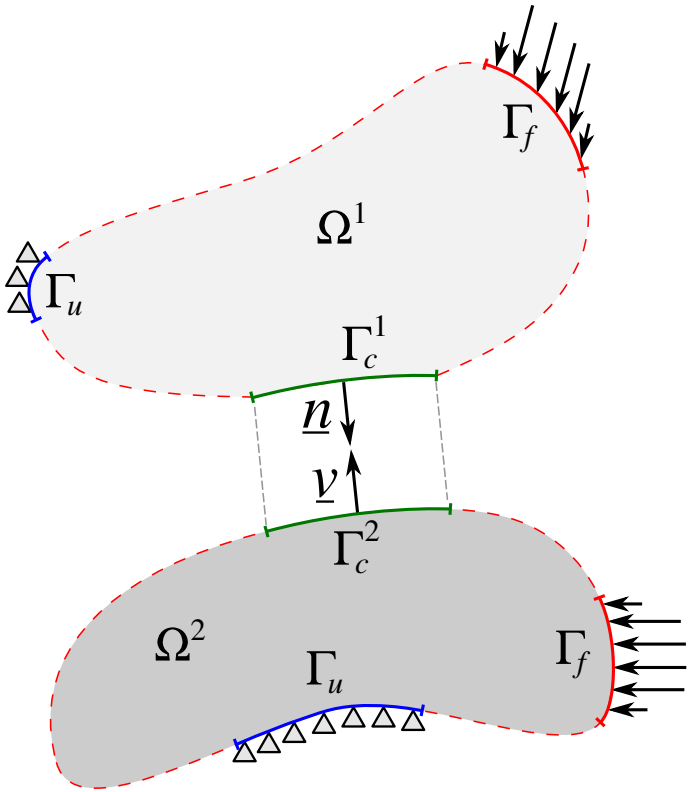


Equilibrium and contact conditions

- **Balance of momentum**

$$\begin{cases} \nabla \cdot \underline{\underline{\sigma}} + \underline{f}_v = 0 & \text{in } \Omega_{1,2} \\ \underline{\underline{\sigma}} \cdot \underline{n} = \underline{t}_0 & \text{on } \Gamma_f \\ \underline{u} = \underline{u}_0 & \text{on } \Gamma_u \\ \quad ? & \text{on } \Gamma_c \end{cases}$$

- **Frictionless contact conditions *(intuitive)***
  1. No penetration
  2. No adhesion
  3. No shear transfer



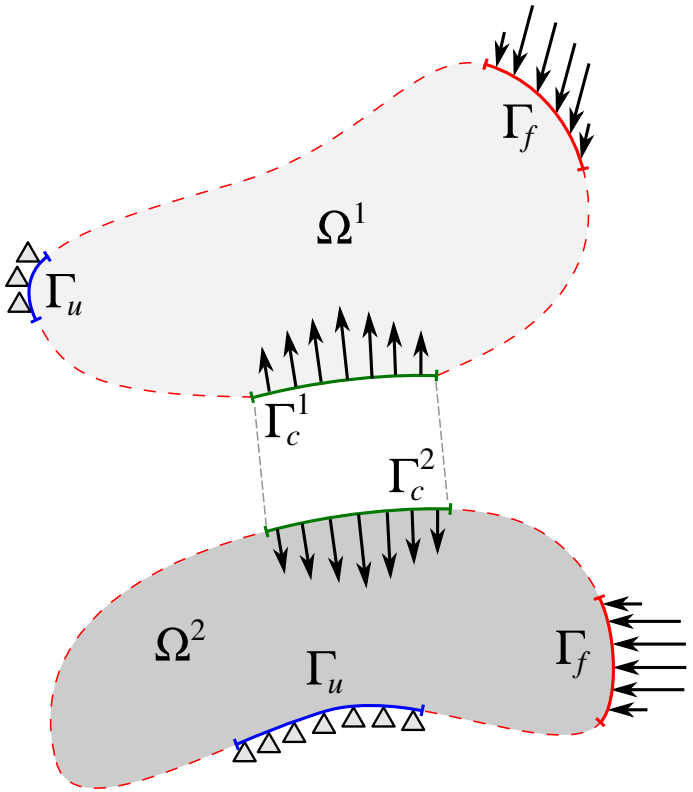
Equilibrium and contact conditions

- **Balance of momentum**

$$\begin{cases} \nabla \cdot \underline{\underline{\sigma}} + \underline{f}_v = 0 & \text{in } \Omega_{1,2} \\ \underline{\underline{\sigma}} \cdot \underline{n} = \underline{t}_0 & \text{on } \Gamma_f \\ \underline{u} = \underline{u}_0 & \text{on } \Gamma_u \\ \quad ? & \text{on } \Gamma_c \end{cases}$$

- **Frictionless contact conditions *(intuitive)***

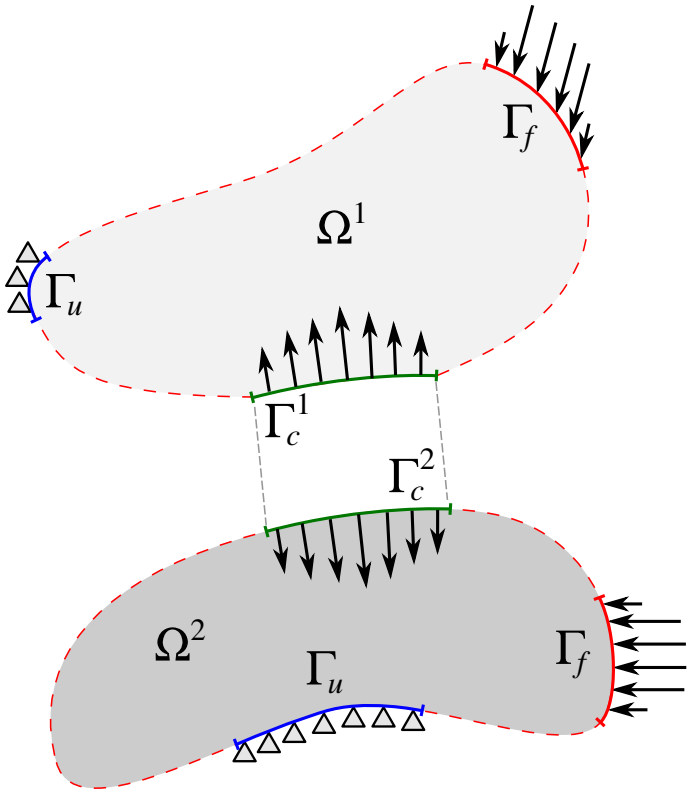
1. No penetration
2. No adhesion
3. No shear transfer



Equilibrium and contact conditions

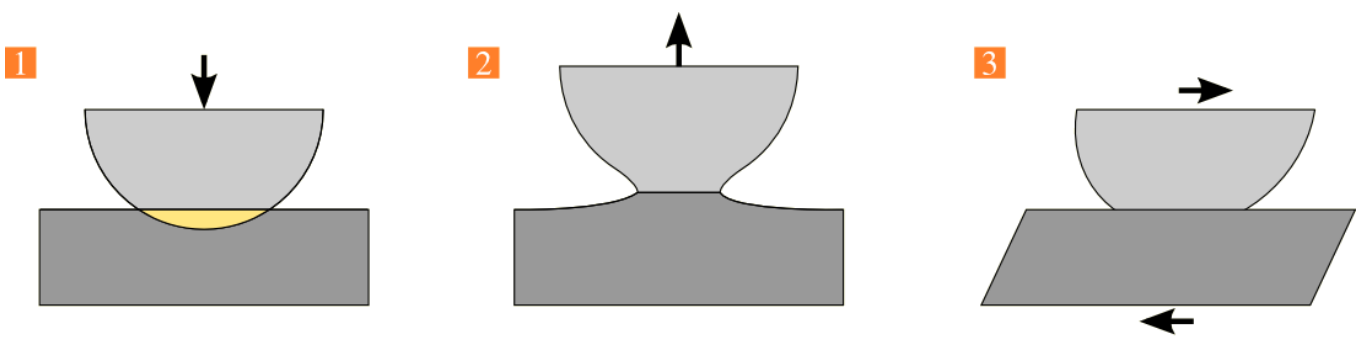
- **Balance of momentum**

$$\begin{cases} \nabla \cdot \underline{\underline{\sigma}} + \underline{f_v} = 0 & \text{in } \Omega_{1,2} \\ \underline{\underline{\sigma}} \cdot \underline{n} = \underline{t}_0 & \text{on } \Gamma_f \\ \underline{u} = \underline{u}_0 & \text{on } \Gamma_u \\ \quad ? & \text{on } \Gamma_c \end{cases}$$



- **Frictionless contact conditions *(intuitive)***

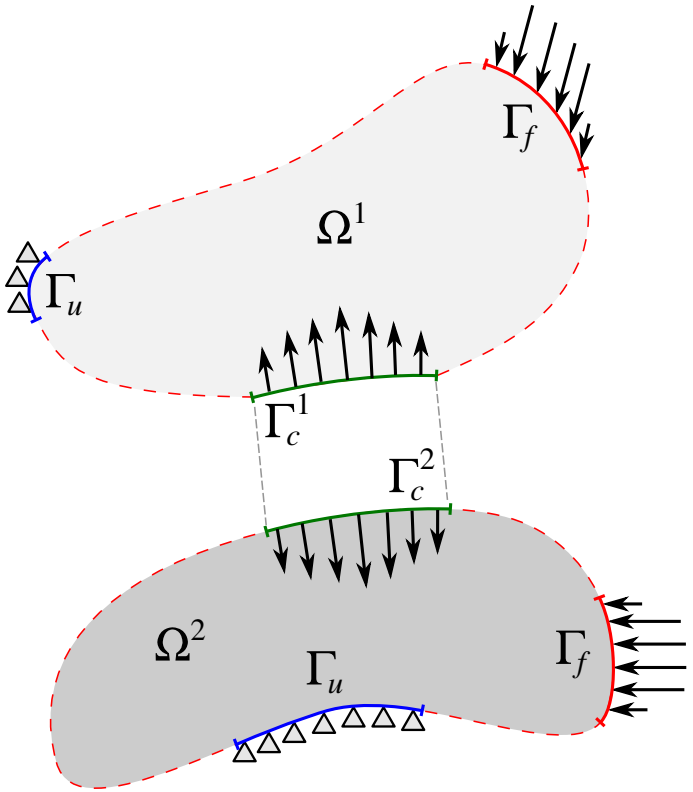
1. No penetration
2. No adhesion
3. No shear transfer



Equilibrium and contact conditions

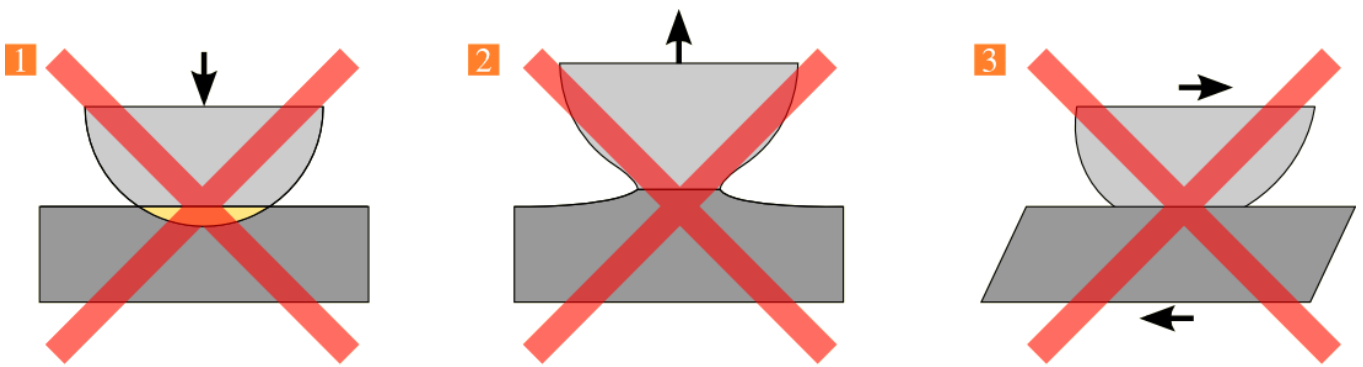
- **Balance of momentum**

$$\begin{cases} \nabla \cdot \underline{\underline{\sigma}} + \underline{f}_v = 0 & \text{in } \Omega_{1,2} \\ \underline{\underline{\sigma}} \cdot \underline{n} = \underline{t}_0 & \text{on } \Gamma_f \\ \underline{u} = \underline{u}_0 & \text{on } \Gamma_u \\ \quad ? & \text{on } \Gamma_c \end{cases}$$



- **Frictionless contact conditions *(intuitive)***

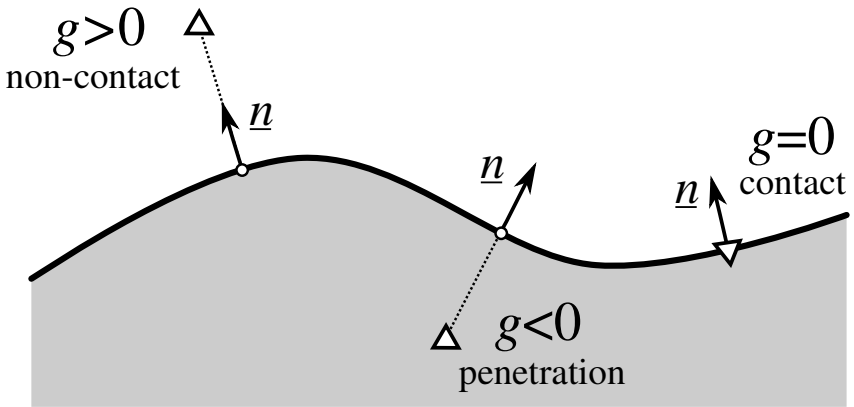
1. No penetration
2. No adhesion
3. No shear transfer



Gap function

- **Gap function** g
  - gap = – penetration
  - asymmetric function
  - defined for
    - separation $g > 0$
    - contact $g = 0$
    - penetration $g < 0$
  - governs normal contact
- **Master and slave split**

  Gap function is determined for all slave points with respect to the master surface



Gap between a slave point and a master surface

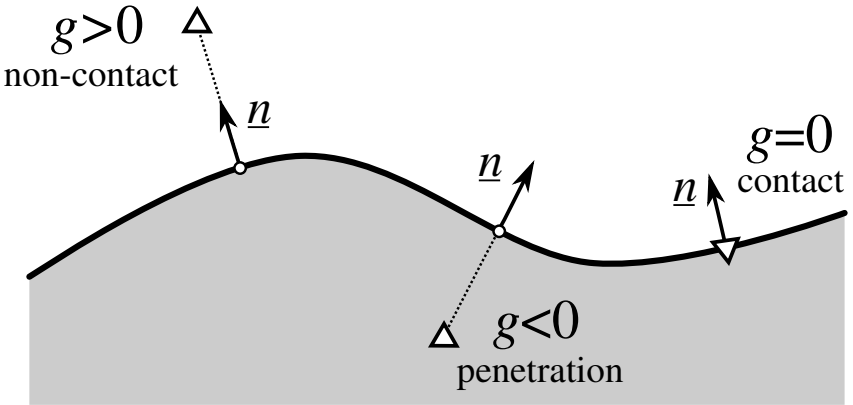
Gap function

- **Gap function** g
  - gap = – penetration
  - asymmetric function
  - defined for
    - separation $g > 0$
    - contact $g = 0$
    - penetration $g < 0$
  - governs normal contact
- **Master and slave split**

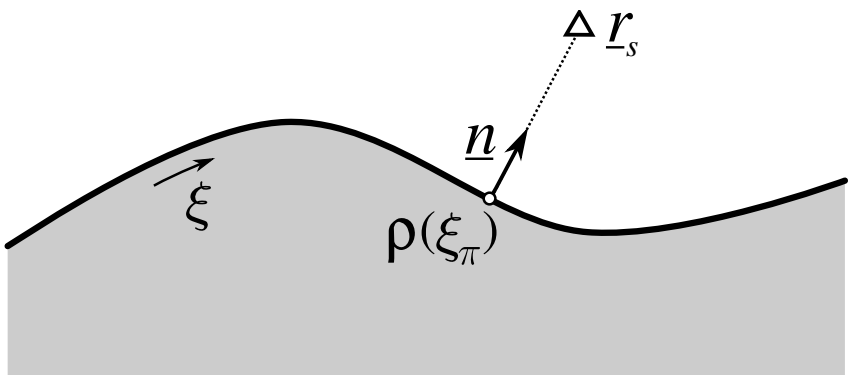
  Gap function is determined for all slave points with respect to the master surface
- **Normal gap**

$$g_n = \underline{n} \cdot \left[\underline{r}_s - \underline{\rho}(\xi_\pi) \right],$$

$\underline{n}$ is a unit normal vector, $\underline{r}_s$ slave point, $\underline{\rho}(\xi_\pi)$ projection point at master surface



Gap between a slave point and a master surface



Definition of the normal gap

Consider existence and uniqueness 

Frictionless or normal contact conditions

- **No penetration**
  Always non-negative gap

  $$g \geq 0$$

- **No adhesion**
  Always non-positive contact pressure

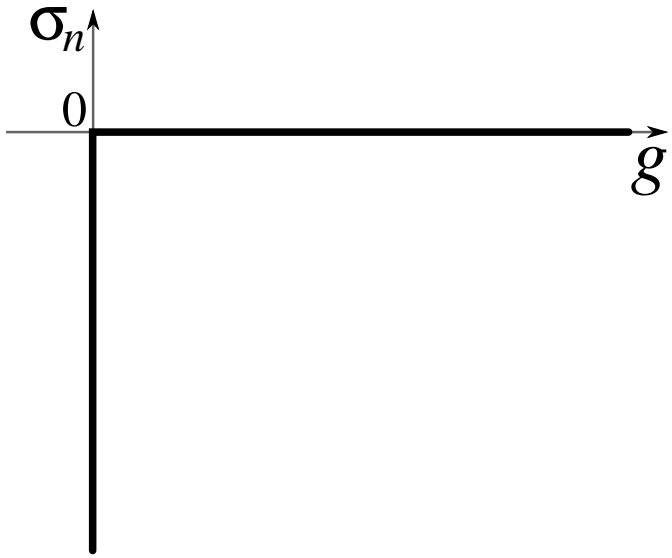
  $$\sigma_n^* \leq 0$$

- **Complementary condition**
  Either zero gap and non-zero pressure, or non-zero gap and zero pressure

  $$g\, \sigma_n = 0$$

- **No shear transfer** *(automatically)*

  $$\underline{\boldsymbol{\sigma}}_t^{**} = 0$$



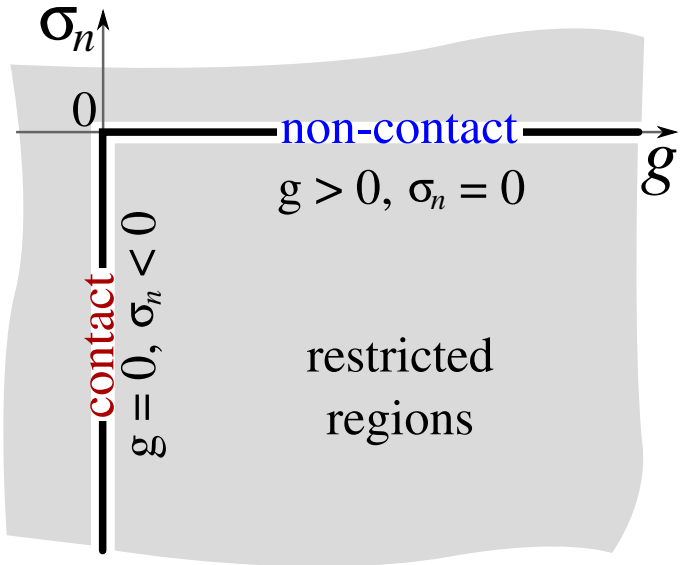
Scheme explaining normal contact conditions

$$\sigma_n^* = (\underline{\underline{\sigma}} \cdot \underline{n}) \cdot \underline{n} = \underline{\underline{\sigma}} : (\underline{n} \otimes \underline{n})$$

$$\underline{\boldsymbol{\sigma}}_t^{**} = \underline{\underline{\sigma}} \cdot \underline{n} - \sigma_n \underline{n} = \underline{n} \cdot \underline{\underline{\sigma}} \cdot \left(\underline{\underline{I}} - \underline{n} \otimes \underline{n}\right)$$

Frictionless or normal contact conditions

- **No penetration**
  Always non-negative gap
  $$g \geq 0$$
- **No adhesion**
  Always non-positive contact pressure
  $$\sigma_n^* \leq 0$$
- **Complementary condition**
  Either zero gap and non-zero pressure, or non-zero gap and zero pressure
  $$g\, \sigma_n = 0$$
- **No shear transfer** *(automatically)*
  $$\underline{\boldsymbol{\sigma}}_t^{**} = 0$$



Improved scheme explaining normal contact conditions

$$\sigma_n^* = (\underline{\underline{\sigma}} \cdot \underline{n}) \cdot \underline{n} = \underline{\underline{\sigma}} : (\underline{n} \otimes \underline{n})$$

$$\underline{\boldsymbol{\sigma}}_t^{**} = \underline{\underline{\sigma}} \cdot \underline{n} - \sigma_n \underline{n} = \underline{n} \cdot \underline{\underline{\sigma}} \cdot \left(\underline{\underline{I}} - \underline{n} \otimes \underline{n}\right)$$

Frictionless or normal contact conditions

In **mechanics:**

Normal contact conditions

≡

Frictionless contact conditions

≡

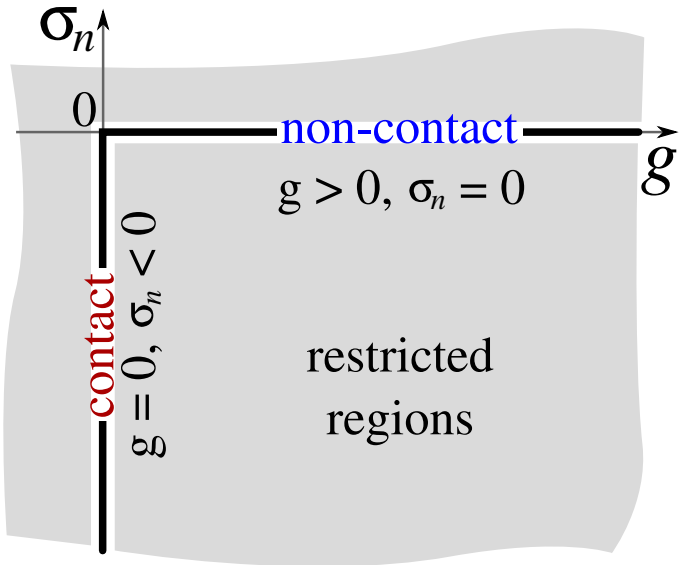
Hertz[1]*-Signorini*[2] *conditions*

≡

Hertz[1]*-Signorini*[2]*-Moreau*[3] *conditions*

also known in **optimization theory** as

Karush[4]*-Kuhn*[5]*-Tucker*[6] *conditions*



Improved scheme explaining normal contact conditions

$$g \geq 0, \qquad \sigma_n \leq 0, \qquad g\sigma_n = 0$$

[1] **Heinrich Rudolf Hertz** (1857–1894) a German physicist who first formulated and solved the frictionless contact problem between elastic ellipsoidal bodies.

[2] **Antonio Signorini** (1888–1963) an Italian mathematical physicist who gave a general and rigorous mathematical formulation of contact constraints.

[3] **Jean Jacques Moreau** (1923) a French mathematician who formulated a non-convex optimization problem based on these conditions and introduced pseudo-potentials in contact mechanics.

[4] **William Karush** (1917–1997), [5] **Harold William Kuhn** (1925) American mathematicians,

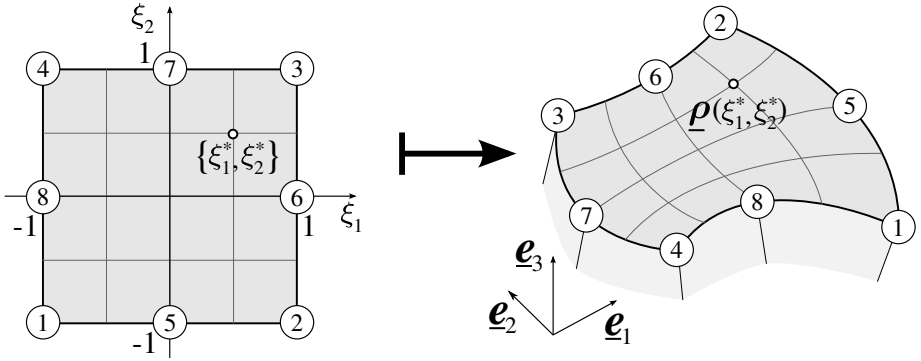
[6] **Albert William Tucker** (1905–1995) a Canadian mathematician.

Relative sliding

Recall:

- Convective coordinate in parent space $\xi_i \in (-1;1)$
- Mapping to real space

$$\underline{\rho}(\xi_1, \xi_2, t) = \sum_{i=1}^{8} N^i(\xi_1, \xi_2)\underline{\rho}^i(t)$$

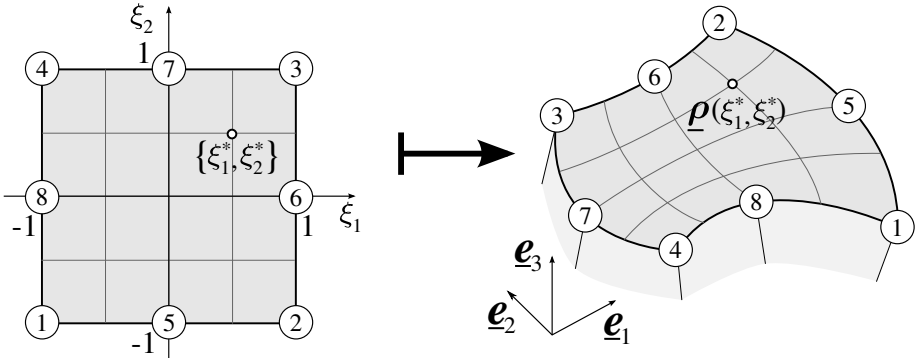


Relative sliding

Recall:

- Convective coordinate in parent space $\xi_i \in (-1;1)$
- Mapping to real space

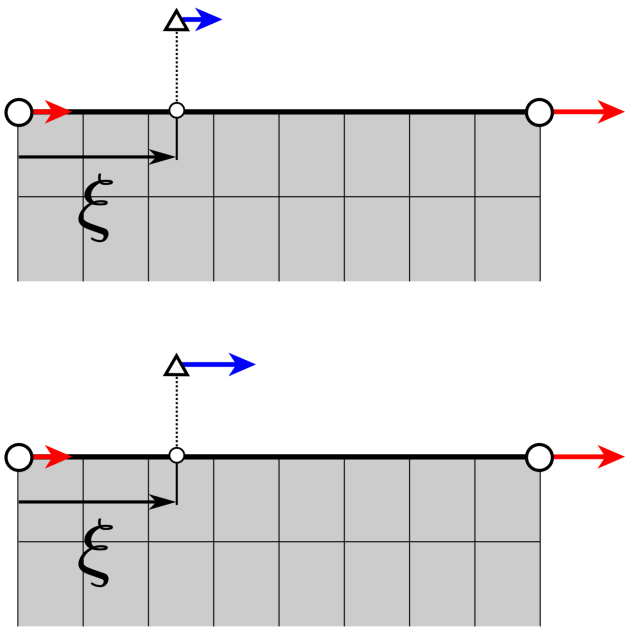
$$\underline{\rho}(\xi_1,\xi_2,t) = \sum_{i=1}^{8} N^i(\xi_1,\xi_2)\underline{\rho}^i(t)$$



- **Tangential slip velocity** $\underline{v}_t$ must take into account:
  - only tangential component
  - relative rigid body motion
  - master's deformation

$$\underline{v}_t = \frac{\partial \underline{\rho}}{\partial \xi_1}\dot{\xi}_1 + \frac{\partial \underline{\rho}}{\partial \xi_2}\dot{\xi}_2$$

where $\partial\underline{\rho}/\partial\xi_i$ are the tangent vectors of the local basis and ξ_i are the convective coordinates.



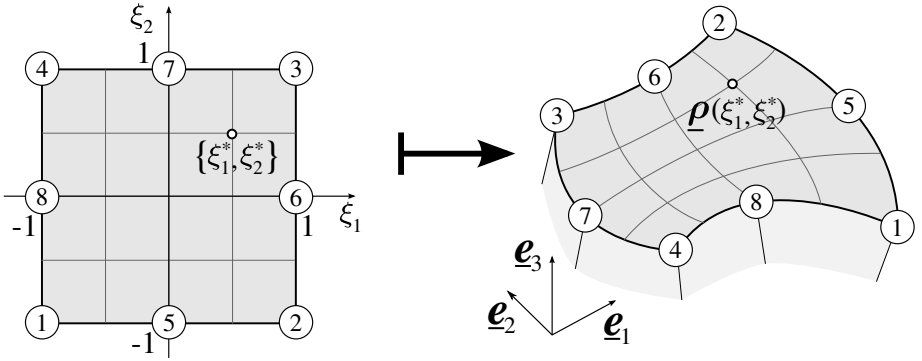
Relative slip between a slave point and a deformable master surface

Relative sliding

Recall:

- Convective coordinate in parent space $\xi_i \in (-1;1)$
- Mapping to real space

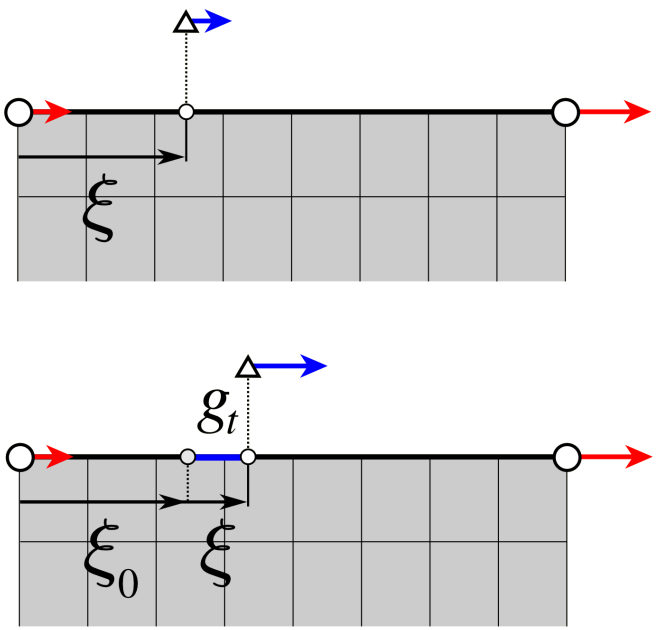
$$\underline{\rho}(\xi_1, \xi_2, t) = \sum_{i=1}^{8} N^i(\xi_1, \xi_2)\underline{\rho}^i(t)$$



- **Tangential slip velocity** $\underline{v}_t$ must take into account:
  - only tangential component
  - relative rigid body motion
  - master's deformation

$$\underline{v}_t = \frac{\partial \underline{\rho}}{\partial \xi_1}\dot{\xi}_1 + \frac{\partial \underline{\rho}}{\partial \xi_2}\dot{\xi}_2$$

where $\partial\underline{\rho}/\partial\xi_i$ are the tangent vectors of the local basis and ξ_i are the convective coordinates.



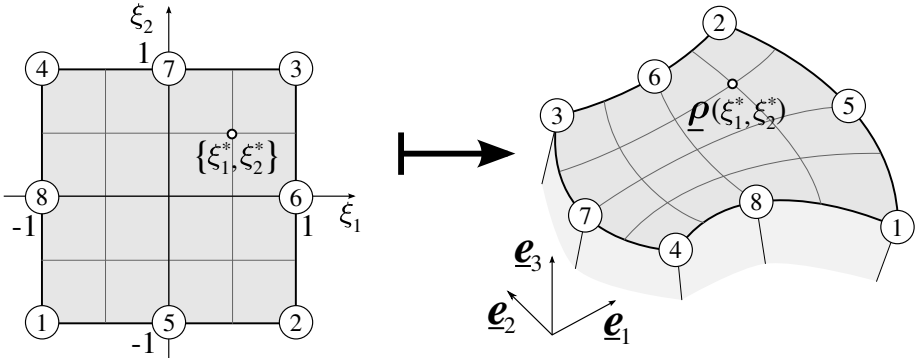
Relative slip between a slave point and a deformable master surface

Relative sliding

Recall:

- Convective coordinate in parent space $\xi_i \in (-1;1)$
- Mapping to real space

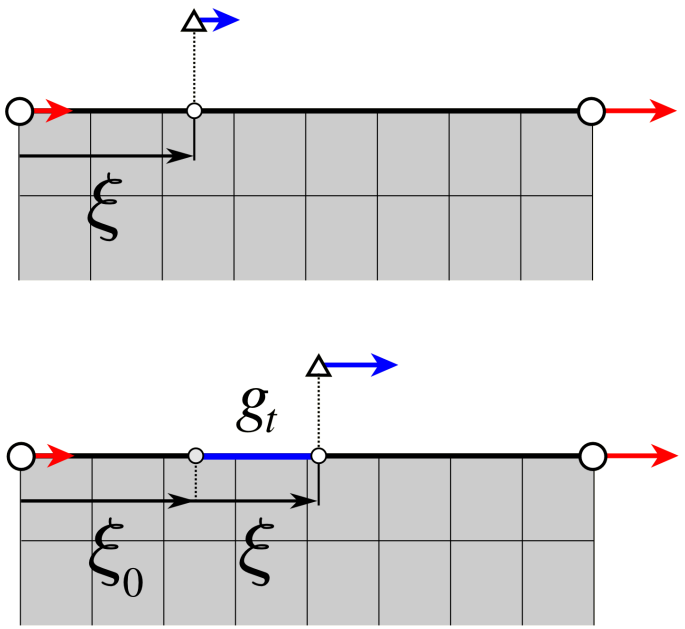
$$\underline{\rho}(\xi_1,\xi_2,t) = \sum_{i=1}^{8} N^i(\xi_1,\xi_2)\underline{\rho}^i(t)$$



- **Tangential slip velocity** $\underline{v}_t$ must take into account:
  - only tangential component
  - relative rigid body motion
  - master's deformation

$$\underline{v}_t = \frac{\partial \underline{\rho}}{\partial \xi_1}\dot{\xi}_1 + \frac{\partial \underline{\rho}}{\partial \xi_2}\dot{\xi}_2$$

where $\partial\underline{\rho}/\partial\xi_i$ are the tangent vectors of the local basis and ξ_i are the convective coordinates.



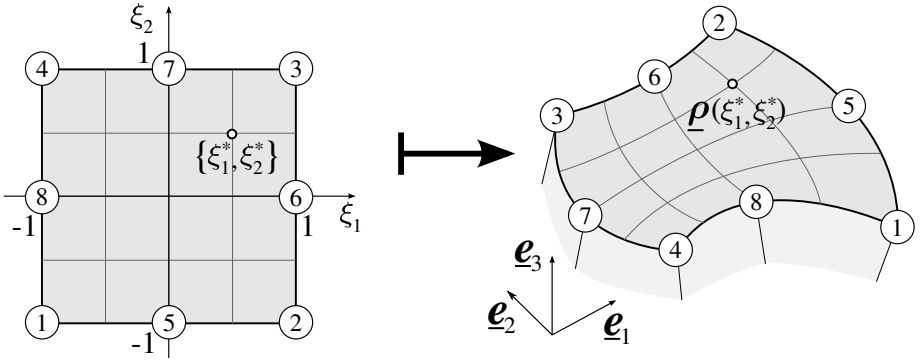
Relative slip between a slave point and a deformable master surface

Relative sliding

Recall:

- Convective coordinate in parent space $\xi_i \in (-1;1)$
- Mapping to real space

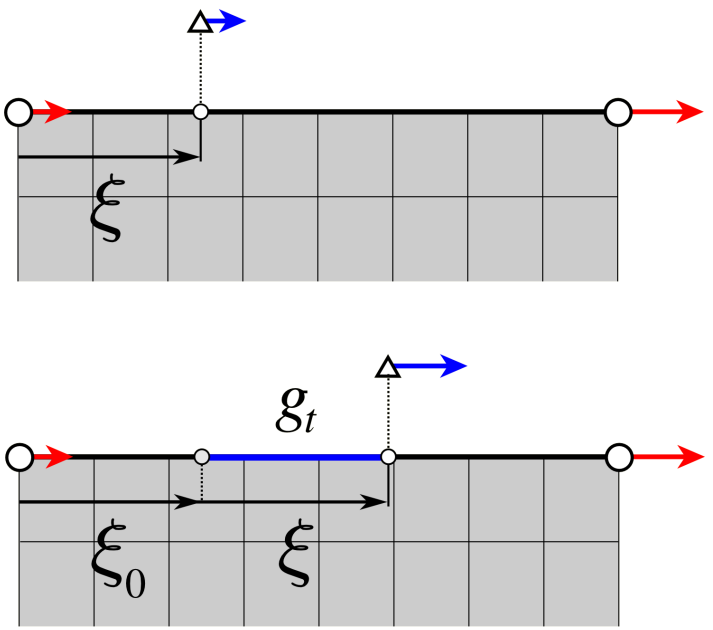
$$\underline{\rho}(\xi_1,\xi_2,t) = \sum_{i=1}^{8} N^i(\xi_1,\xi_2)\underline{\rho}^i(t)$$



- **Tangential slip velocity** $\underline{v}_t$ must take into account:
  - only tangential component
  - relative rigid body motion
  - master's deformation

$$\underline{v}_t = \frac{\partial \underline{\rho}}{\partial \xi_1}\dot{\xi}_1 + \frac{\partial \underline{\rho}}{\partial \xi_2}\dot{\xi}_2$$

where $\partial\underline{\rho}/\partial\xi_i$ are the tangent vectors of the local basis and ξ_i are the convective coordinates.



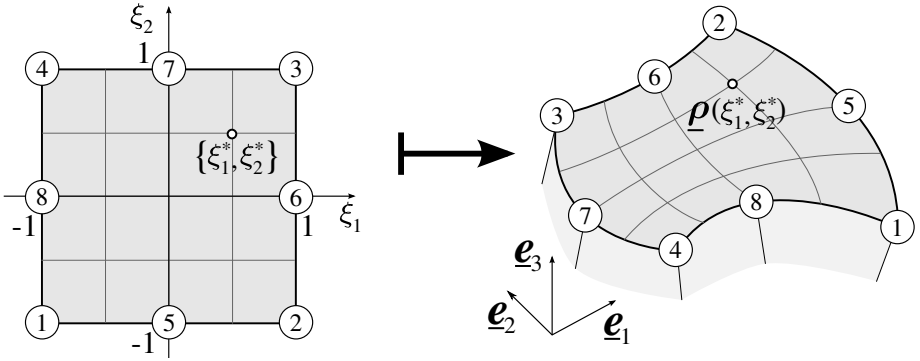
Relative slip between a slave point and a deformable master surface

Relative sliding

Recall:

- Convective coordinate in parent space $\xi_i \in (-1;1)$
- Mapping to real space

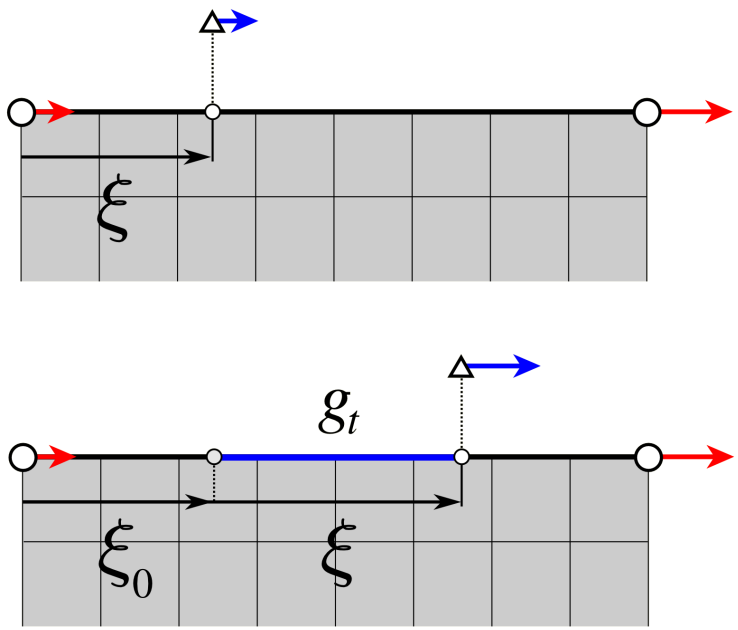
$$\underline{\rho}(\xi_1,\xi_2,t) = \sum_{i=1}^{8} N^i(\xi_1,\xi_2)\underline{\rho}^i(t)$$



- **Tangential slip velocity** $\underline{v}_t$ must take into account:
  - only tangential component
  - relative rigid body motion
  - master's deformation

$$\underline{v}_t = \frac{\partial \underline{\rho}}{\partial \xi_1}\dot{\xi}_1 + \frac{\partial \underline{\rho}}{\partial \xi_2}\dot{\xi}_2$$

where $\partial\underline{\rho}/\partial\xi_i$ are the tangent vectors of the local basis and ξ_i are the convective coordinates.



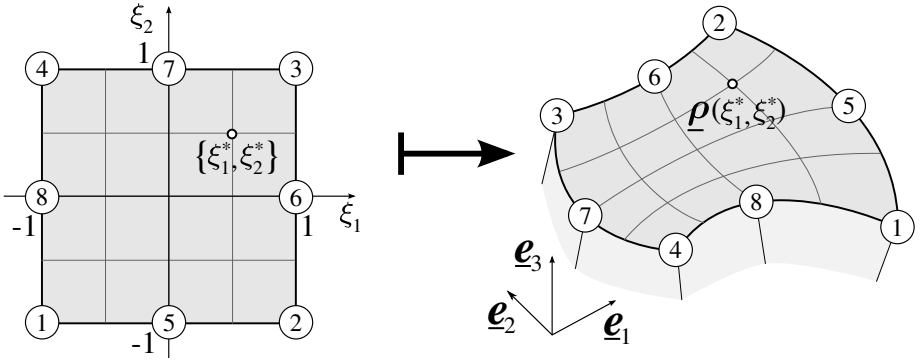
Relative slip between a slave point and a deformable master surface

Relative sliding

Recall:

- Convective coordinate in parent space $\xi_i \in (-1;1)$
- Mapping to real space

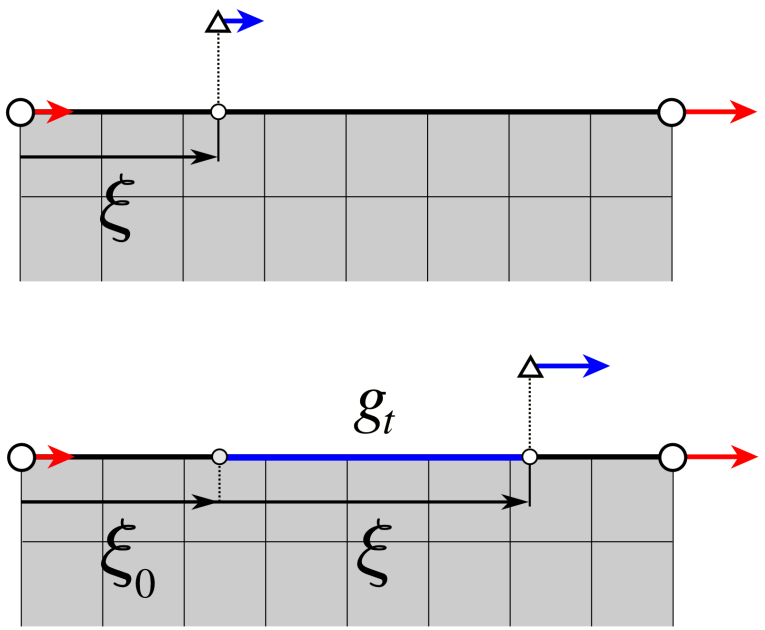
$$\underline{\rho}(\xi_1,\xi_2,t) = \sum_{i=1}^{8} N^i(\xi_1,\xi_2)\underline{\rho}^i(t)$$



- **Tangential slip velocity** $\underline{v}_t$ must take into account:
  - only tangential component
  - relative rigid body motion
  - master's deformation

$$\underline{v}_t = \frac{\partial \underline{\rho}}{\partial \xi_1}\dot{\xi}_1 + \frac{\partial \underline{\rho}}{\partial \xi_2}\dot{\xi}_2$$

where $\partial\underline{\rho}/\partial\xi_i$ are the tangent vectors of the local basis and ξ_i are the convective coordinates.



Relative slip between a slave point and a deformable master surface

Relative sliding: example

Consider a one-dimensional example:
P is a projection of A on segment BC.

$$x_P = \xi x_C + (1-\xi)x_B \qquad (1)$$

Velocity of the projection point

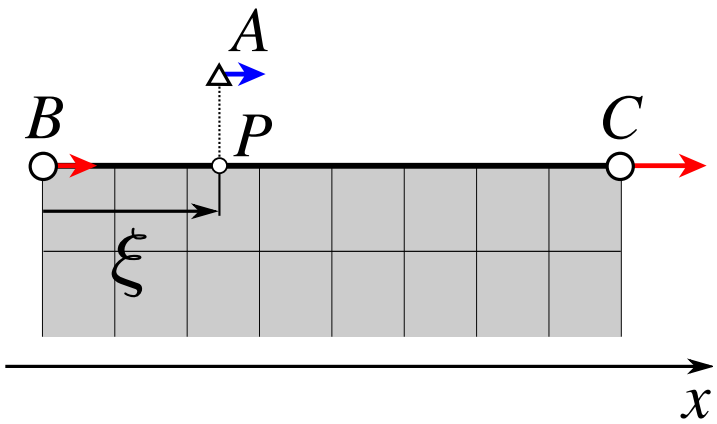
$$\dot{x}_P = \underbrace{\xi \dot{x}_C + (1-\xi)\dot{x}_B}_{\frac{\partial x_P}{\partial t}} + \underbrace{(x_C - x_B)\dot{\xi}}_{\frac{\partial x_P}{\partial \xi}\dot{\xi}}$$

Substract the velocity of point x_P for fixed ξ

$$v_t = \dot{x}_P - \frac{\partial x_P}{\partial t} = (x_C - x_B)\dot{\xi} = \frac{\partial x}{\partial \xi}\dot{\xi}$$

Compute tangential slip increment

$$\Delta g_t^{n+1} = \left.\frac{\partial x}{\partial \xi}\right|_{\xi^n} (\xi^{n+1} - \xi^n)$$



Example of a one-dimensional relative slip

Relative sliding: example

Consider a one-dimensional example:
P is a projection of A on segment BC.

$$x_P = \xi x_C + (1-\xi)x_B \qquad (1)$$

Velocity of the projection point

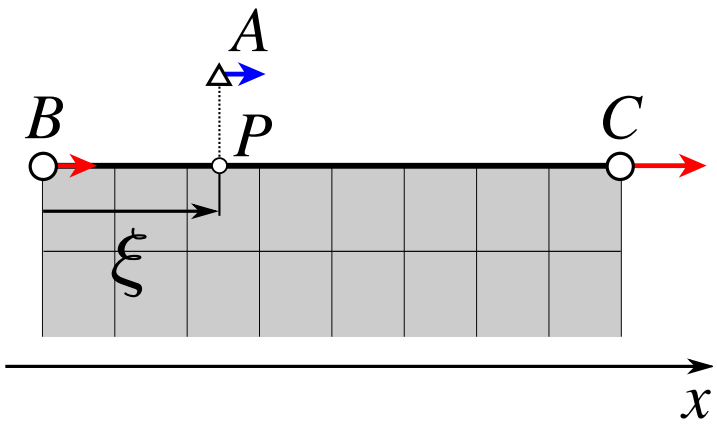
$$\dot{x}_P = \underbrace{\xi\dot{x}_C + (1-\xi)\dot{x}_B}_{\frac{\partial x_P}{\partial t}} + \underbrace{(x_C - x_B)\dot{\xi}}_{\frac{\partial x_P}{\partial \xi}\dot{\xi}}$$

Substract the velocity of point x_P for fixed ξ

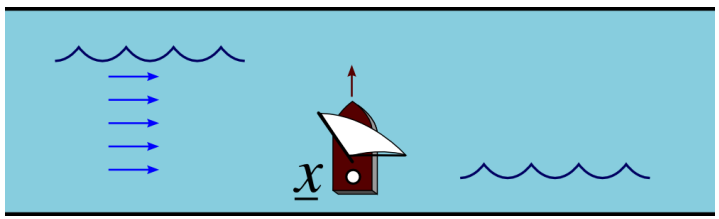
$$v_t = \dot{x}_P - \frac{\partial x_P}{\partial t} = (x_C - x_B)\dot{\xi} = \frac{\partial x}{\partial \xi}\dot{\xi}$$

Compute tangential slip increment

$$\Delta g_t^{n+1} = \left.\frac{\partial x}{\partial \xi}\right|_{\xi^n} (\xi^{n+1} - \xi^n)$$



Example of a one-dimensional relative slip



Ship-river analogy

Relative sliding: example

Consider a one-dimensional example:
P is a projection of A on segment BC.

$$x_P = \xi x_C + (1-\xi)x_B \qquad (1)$$

Velocity of the projection point

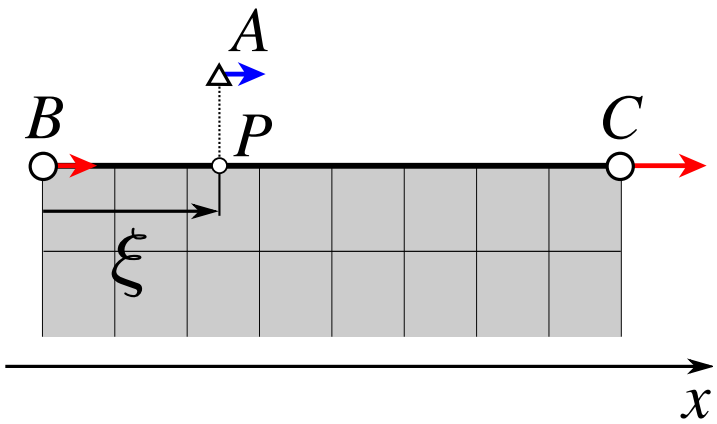
$$\dot{x}_P = \underbrace{\xi\dot{x}_C + (1-\xi)\dot{x}_B}_{\frac{\partial x_P}{\partial t}} + \underbrace{(x_C - x_B)\dot{\xi}}_{\frac{\partial x_P}{\partial \xi}\dot{\xi}}$$

Substract the velocity of point x_P for fixed ξ

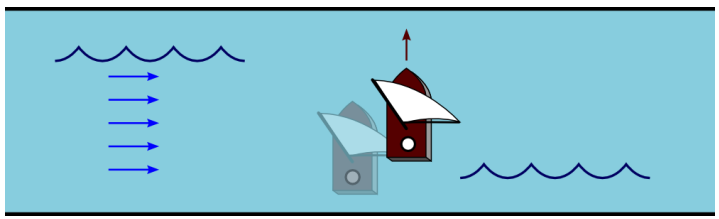
$$v_t = \dot{x}_P - \frac{\partial x_P}{\partial t} = (x_C - x_B)\dot{\xi} = \frac{\partial x}{\partial \xi}\dot{\xi}$$

Compute tangential slip increment

$$\Delta g_t^{n+1} = \left.\frac{\partial x}{\partial \xi}\right|_{\xi^n} (\xi^{n+1} - \xi^n)$$



Example of a one-dimensional relative slip



Ship-river analogy

Relative sliding: example

Consider a one-dimensional example:
P is a projection of A on segment BC.

$$x_P = \xi x_C + (1-\xi)x_B \qquad (1)$$

Velocity of the projection point

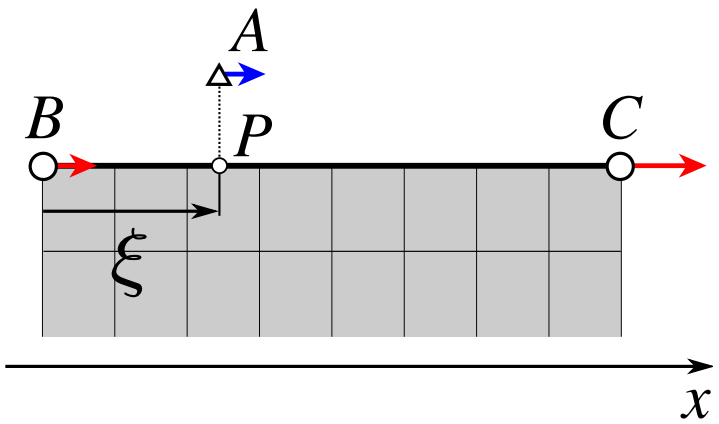
$$\dot{x}_P = \underbrace{\xi\dot{x}_C + (1-\xi)\dot{x}_B}_{\frac{\partial x_P}{\partial t}} + \underbrace{(x_C - x_B)\dot{\xi}}_{\frac{\partial x_P}{\partial \xi}\dot{\xi}}$$

Substract the velocity of point x_P for fixed ξ

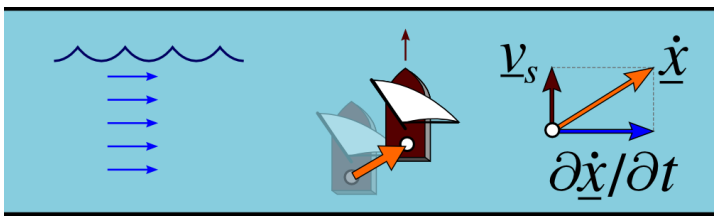
$$v_t = \dot{x}_P - \frac{\partial x_P}{\partial t} = (x_C - x_B)\dot{\xi} = \frac{\partial x}{\partial \xi}\dot{\xi}$$

Compute tangential slip increment

$$\Delta g_t^{n+1} = \left.\frac{\partial x}{\partial \xi}\right|_{\xi^n} (\xi^{n+1} - \xi^n)$$



Example of a one-dimensional relative slip



Ship-river analogy

Li derivative: the change of a vector field along the change of another vector field

Amontons-Coulomb's friction

- **No contact** $g > 0,\ \sigma_n = 0$
- **Stick** $|\underline{v}_t| = 0$

  Inside slip surface/Coulomb's cone

$$f = |\underline{\sigma}_t| - \mu|\sigma_n| < 0$$

- **Slip** $|\underline{v}_t| > 0$

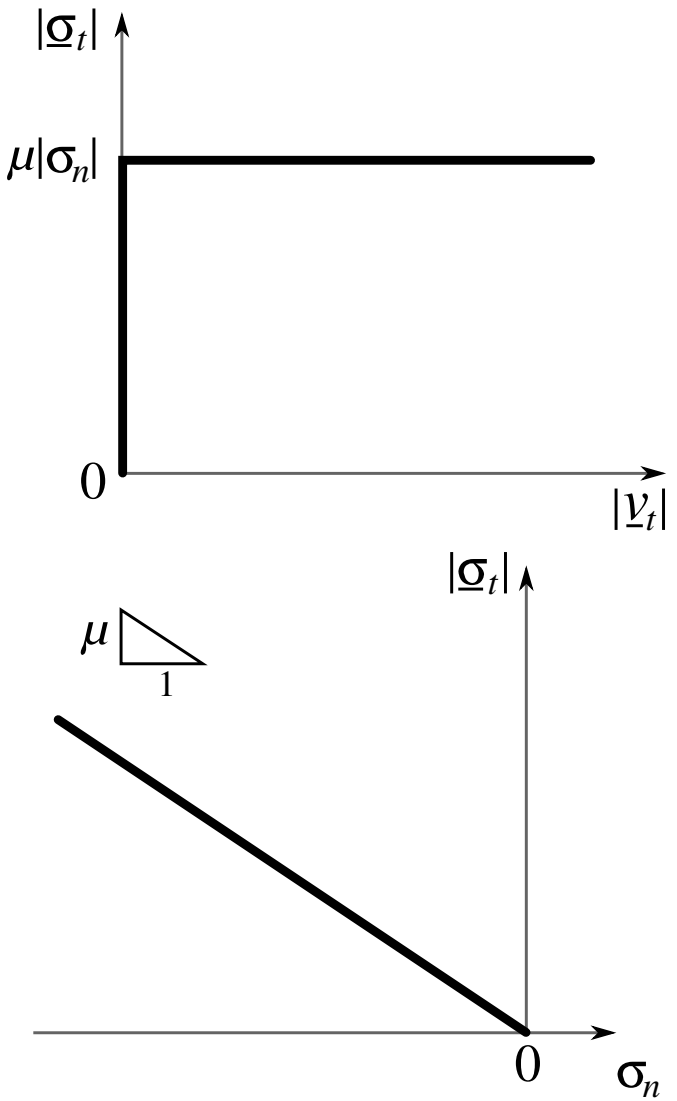
  On slip surface/Coulomb's cone

$$f = |\underline{\sigma}_t| - \mu|\sigma_n| = 0$$

- **Complementary condition**

  Either zero velocity and negative slip criterion, or non-zero velocity and zero slip criterion

$$|\underline{v}_t|\Big(\,|\underline{\sigma}_t| - \mu|\sigma_n|\,\Big) = 0$$



Scheme explaining frictional contact conditions

Amontons-Coulomb's friction

- **No contact** $g > 0,\ \sigma_n = 0$
- **Stick** $|\underline{v}_t| = 0$

  Inside slip surface/Coulomb's cone

$$f = |\underline{\sigma}_t| - \mu|\sigma_n| < 0$$

- **Slip** $|\underline{v}_t| > 0$

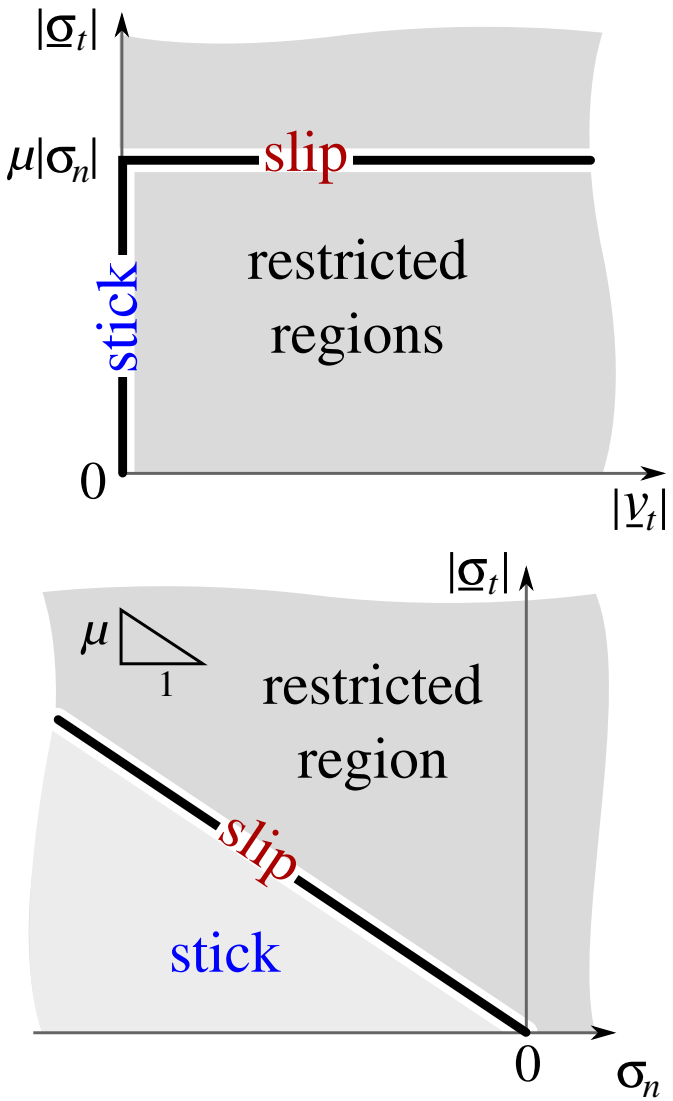
  On slip surface/Coulomb's cone

$$f = |\underline{\sigma}_t| - \mu|\sigma_n| = 0$$

- **Complementary condition**

  Either zero velocity and negative slip criterion, or non-zero velocity and zero slip criterion

$$|\underline{v}_t|\Big(\,|\underline{\sigma}_t| - \mu|\sigma_n|\,\Big) = 0$$



Improved scheme explaining frictional contact conditions

Amontons-Coulomb's friction

- **No contact** $g > 0,\ \sigma_n = 0$
- **Stick** $|\underline{v}_t| = 0$

  Inside slip surface/Coulomb's cone

$$f = |\underline{\sigma}_t| - \mu|\sigma_n| < 0$$

- **Slip** $|\underline{v}_t| > 0$

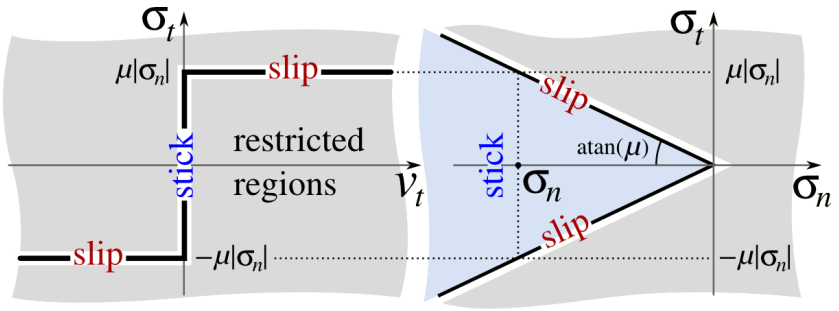
  On slip surface/Coulomb's cone

$$f = |\underline{\sigma}_t| - \mu|\sigma_n| = 0$$

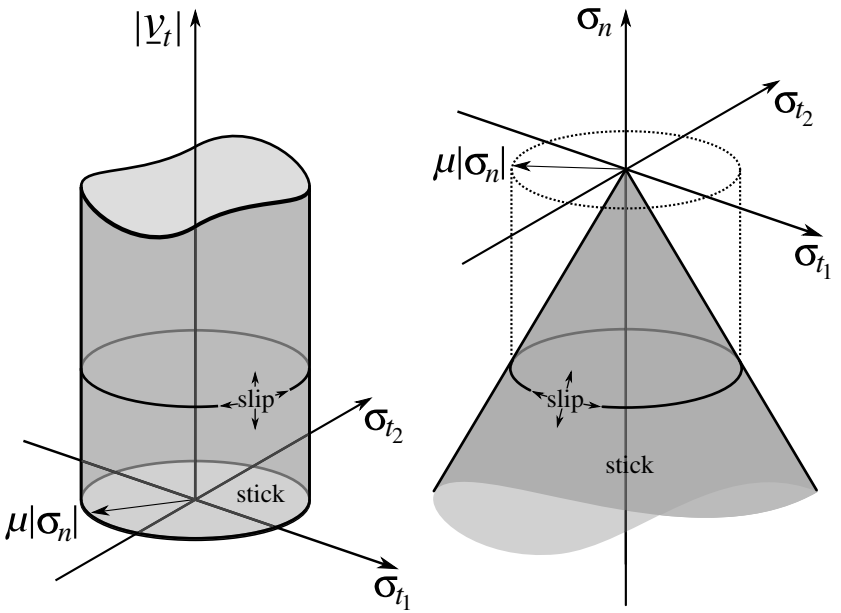
- **Complementary condition**

  Either zero velocity and negative slip criterion, or non-zero velocity and zero slip criterion

$$|\underline{v}_t|\Big(\,|\underline{\sigma}_t| - \mu|\sigma_n|\,\Big) = 0$$



Scheme of 2D frictional contact

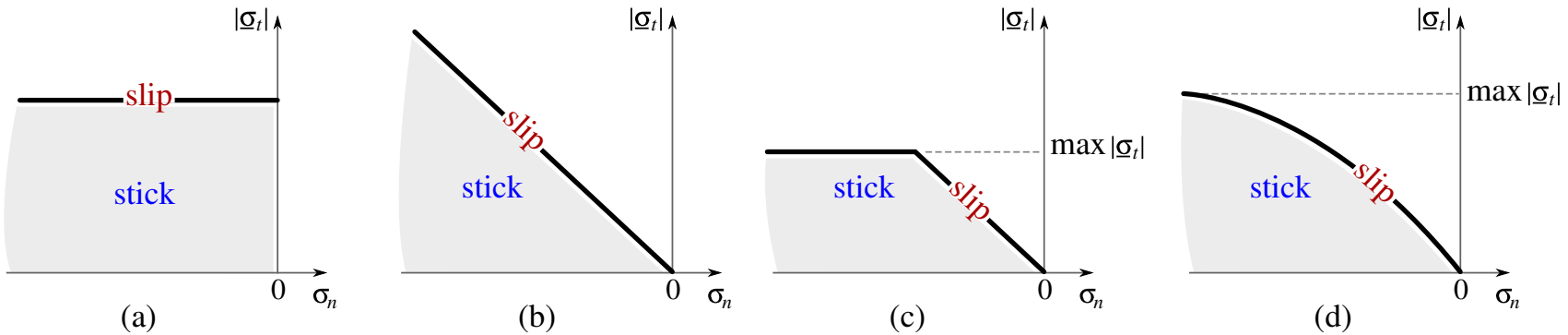


Scheme of 3D frictional contact

$$|\underline{v}_t| \geq 0, \qquad |\underline{\sigma}_t| - \mu|\sigma_n| \leq 0, \qquad |\underline{v}_t|\Big(\,|\underline{\sigma}_t| - \mu|\sigma_n|\,\Big) = 0$$

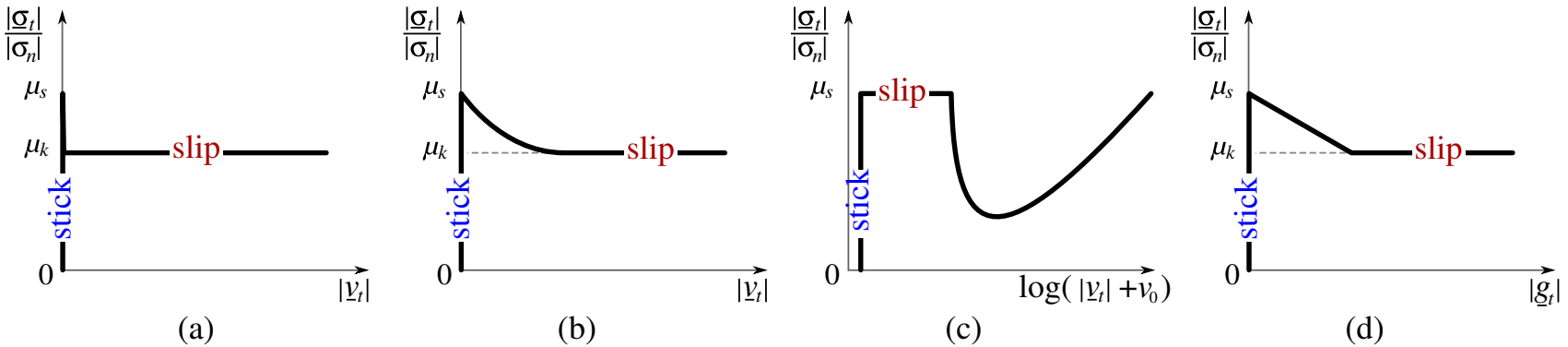
More friction laws

- **Static criteria**



(a) Tresca (b) Amontons-Coulomb (c) Coulomb-Orowan (d) Shaw

- **Kinetic criteria**



(a,b) velocity weakening (c) velocity weakening-strengthening
(d) Linear slip weakening

- μ_s static and μ_k kinetic coefficients of friction.

Rate and state friction and regularization

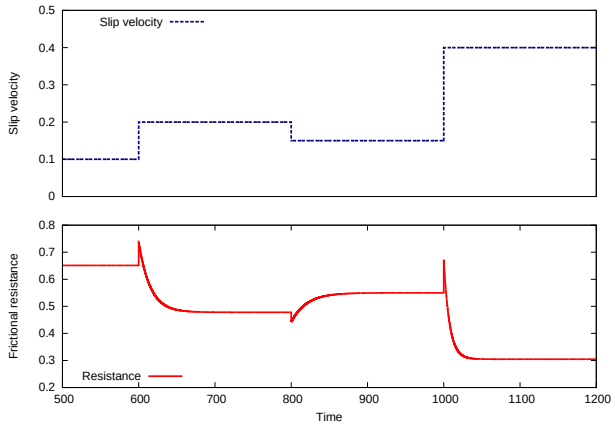
- **Rate and state friction law**
  - Rate $v_t = |\underline{v}_t|$ – relative slip velocity
  - State θ – $\approx$ internal time
  - Dieterich–Ruina–Perrin (1979, 83, 95)

    Frictional resistance

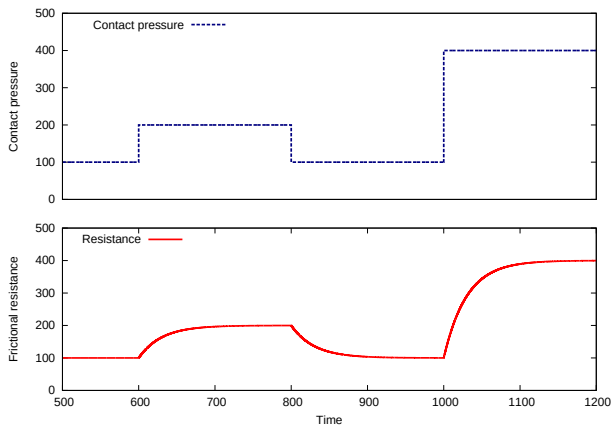
    $$\sigma_t^c = |\sigma_n| \left[\mu_s + b\theta + a \ln(v_t/v_0)\right]$$

    Evolution of the state variable

    $$\dot{\theta} = -\frac{v_t}{L}\left[\theta + \ln\left(\frac{v_t}{v_0}\right)\right]$$
- **Prakash-Clifton friction law (1992,2000)**
  - Viscous type evolution of frictional resistance σ_t
  - $\dot{\sigma}_t = -\frac{v_t}{L}(\sigma_t + \mu\sigma_n)$



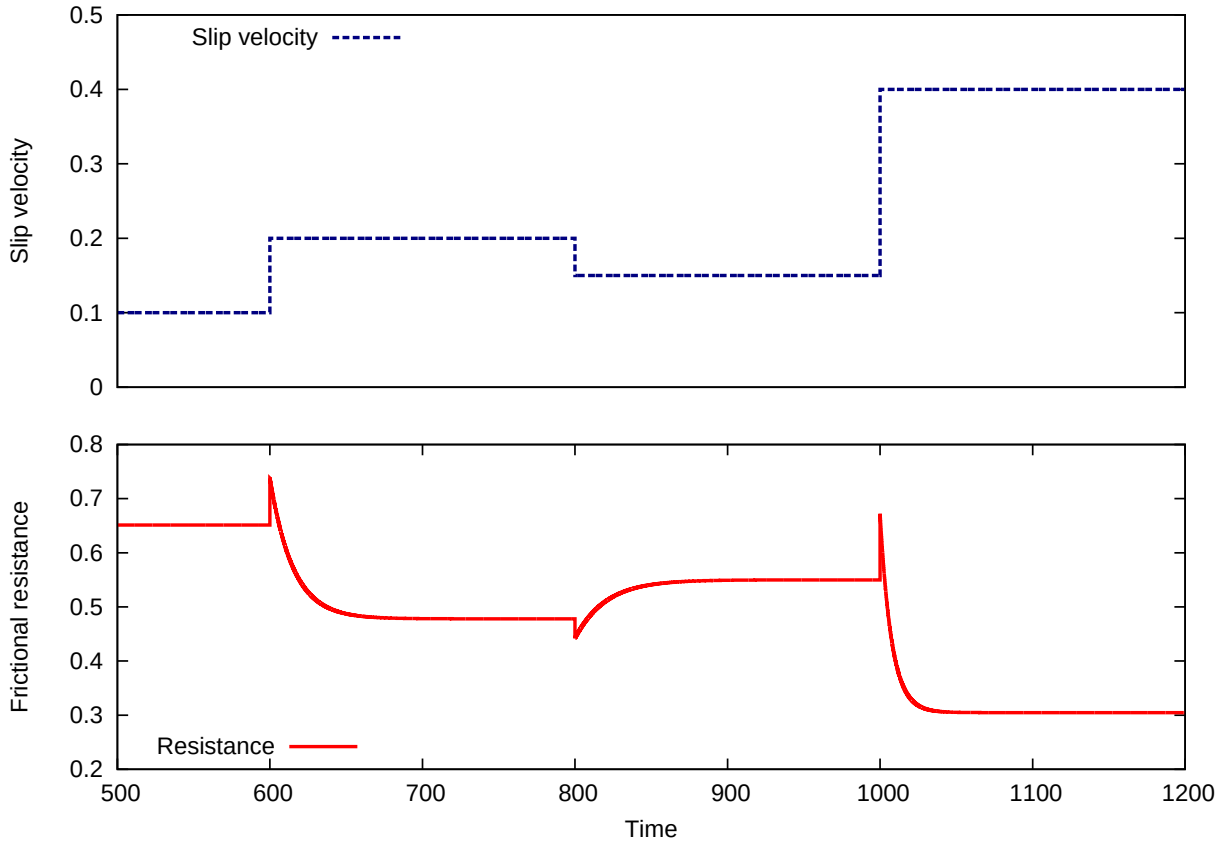
Rate and state friction law



Prakash-Clifton regularization

Rate and state friction and regularization

- **Rate and state friction law**



Rate and state friction and regularization

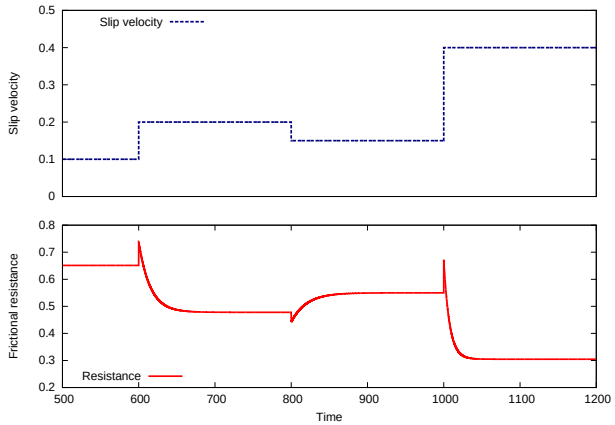
- **Rate and state friction law**
  - Rate $v_t = |\underline{v}_t|$ – relative slip velocity
  - State θ – $\approx$ internal time
  - Dieterich–Ruina–Perrin (1979, 83, 95)

    Frictional resistance

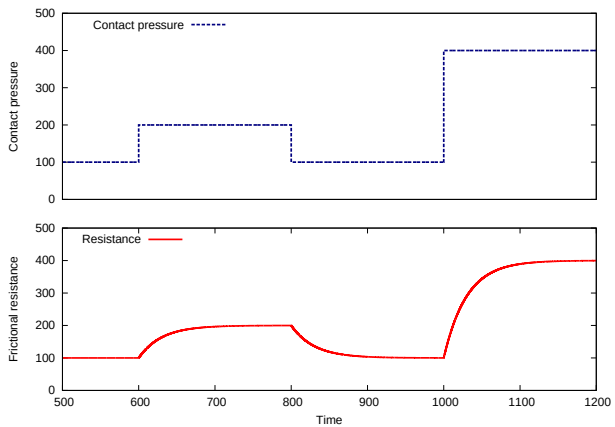
    $$\sigma_t^c = |\sigma_n| \left[\mu_s + b\theta + a \ln(v_t/v_0)\right]$$

    Evolution of the state variable

    $$\dot{\theta} = -\frac{v_t}{L}\left[\theta + \ln\left(\frac{v_t}{v_0}\right)\right]$$
- **Prakash-Clifton friction law (1992,2000)**
  - Viscous type evolution of frictional resistance σ_t
  - $\dot{\sigma}_t = -\frac{v_t}{L}(\sigma_t + \mu\sigma_n)$



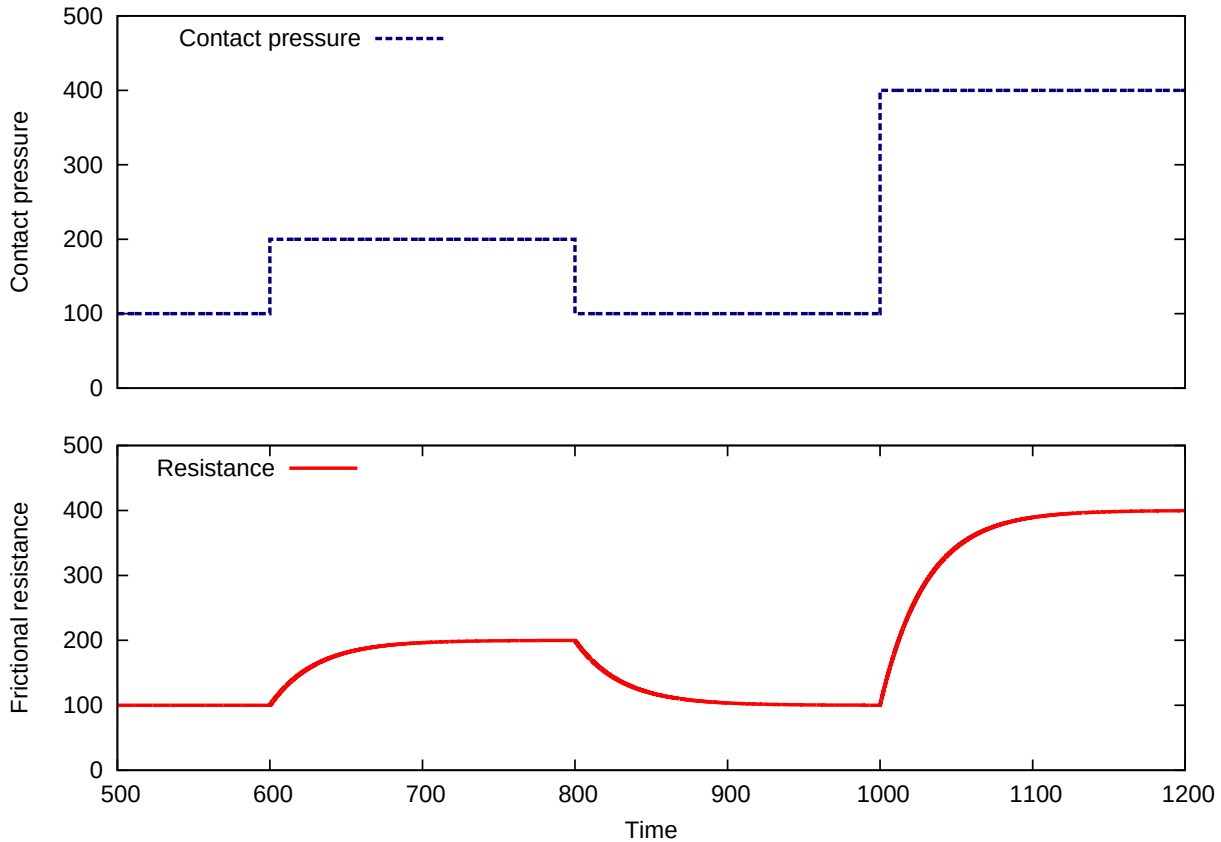
Rate and state friction law



Prakash-Clifton regularization

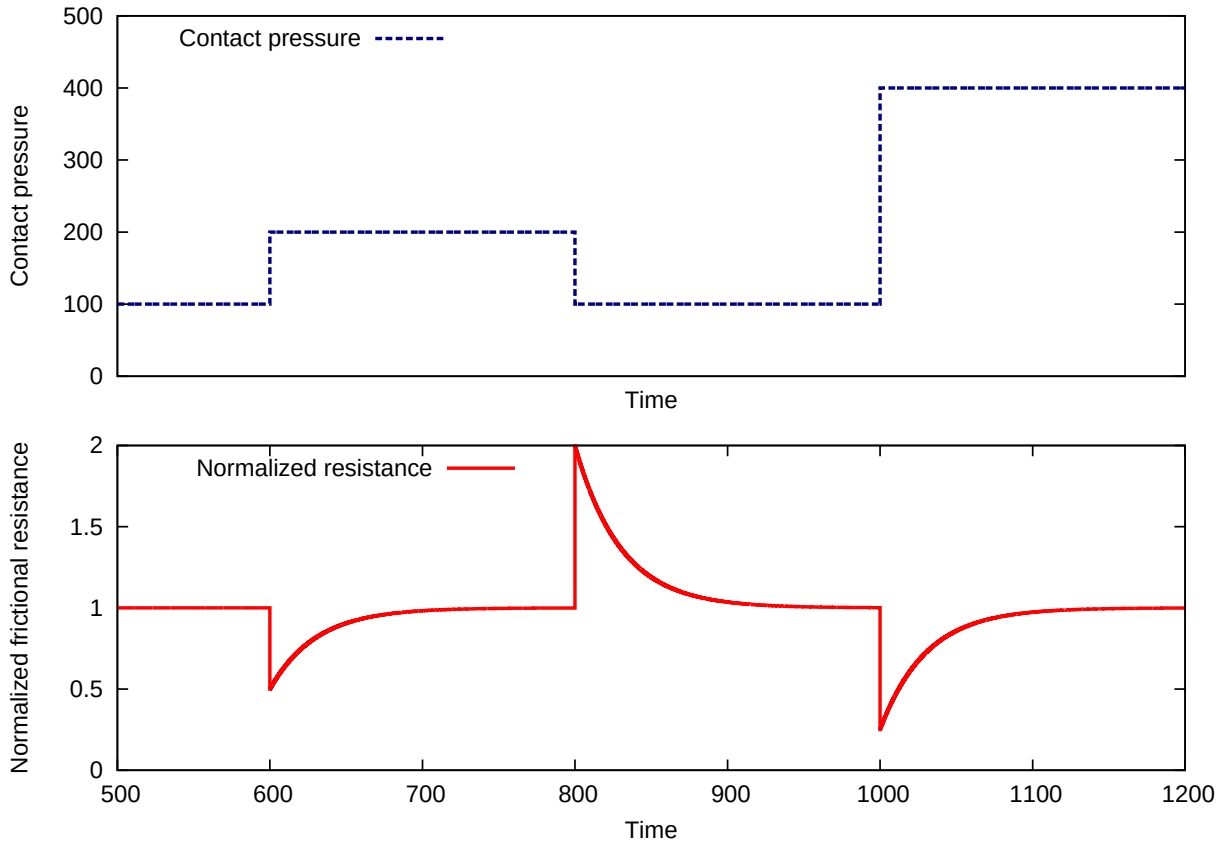
Rate and state friction and regularization

- Prakash-Clifton friction law (1992,2000)



Rate and state friction and regularization

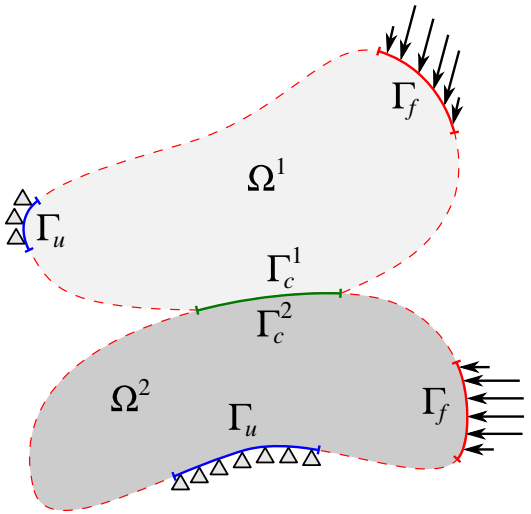
- Prakash-Clifton friction law (1992,2000)



From strong to weak form

- Balance of momentum and boundary conditions

$$\nabla \cdot \underline{\underline{\sigma}} + \underline{f}_v = 0 \quad \text{in } \Omega = \Omega_1 \cup \Omega_2 \qquad + \text{B.C.}$$



Two solids in contact

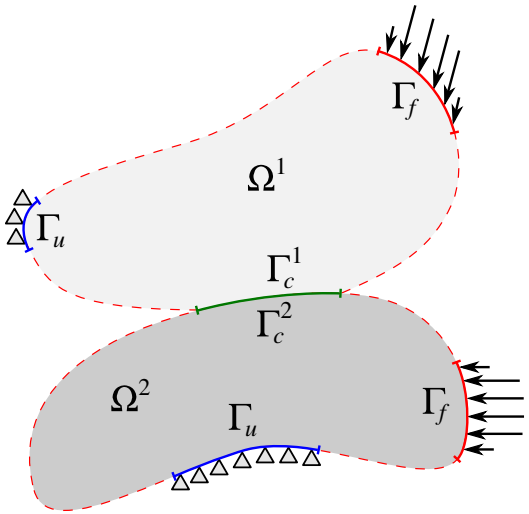
From strong to weak form

- Balance of momentum and boundary conditions

$$\nabla \cdot \underline{\underline{\sigma}} + \underline{f}_v = 0 \quad \text{in } \Omega = \Omega_1 \cup \Omega_2 \qquad + \text{B.C.}$$

- Balance of virtual works 

$$\boxed{\int_{\partial\Omega} \underline{n} \cdot \underline{\underline{\sigma}} \cdot \delta\underline{u}\, d\Gamma} + \int_{\Omega} \left[\underline{f}_v \cdot \delta\underline{u} - \underline{\underline{\sigma}} \cdot\cdot\, \delta\nabla\underline{u}\right] d\Omega = 0$$



Two solids in contact

From strong to weak form

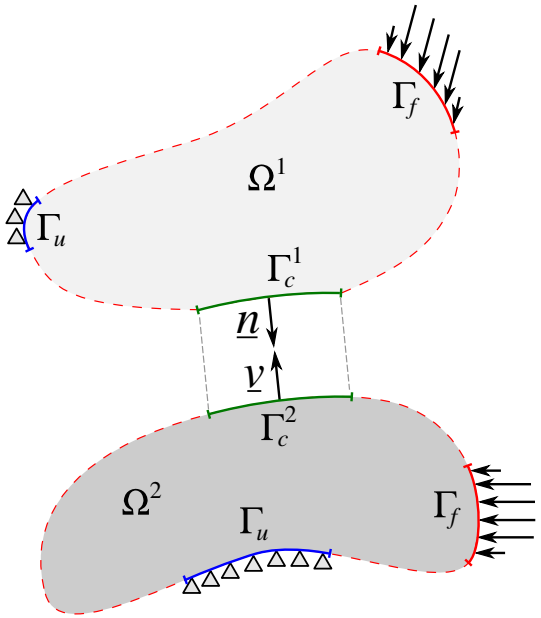
- Balance of momentum and boundary conditions

$$\nabla \cdot \underline{\underline{\sigma}} + \underline{f}_v = 0 \quad \text{in } \Omega = \Omega_1 \cup \Omega_2 \qquad + \text{B.C.}$$

- Balance of virtual works 

$$\boxed{\int_{\partial\Omega} \underline{n} \cdot \underline{\underline{\sigma}} \cdot \delta\underline{u}\, d\Gamma} =$$

$$\boxed{\int_{\overline{\Gamma}_c^1} \underline{n} \cdot \underline{\underline{\sigma}} \cdot \delta\underline{\rho}\, d\overline{\Gamma}_c^1 + \int_{\overline{\Gamma}_c^2} \underline{\nu} \cdot \underline{\underline{\sigma}} \cdot \delta\underline{r}\, d\overline{\Gamma}_c^2} + \int_{\Gamma_f} \underline{\sigma}_0 \cdot \delta\underline{u}\, d\Gamma_f$$



Two solids in contact

From strong to weak form

- Balance of momentum and boundary conditions

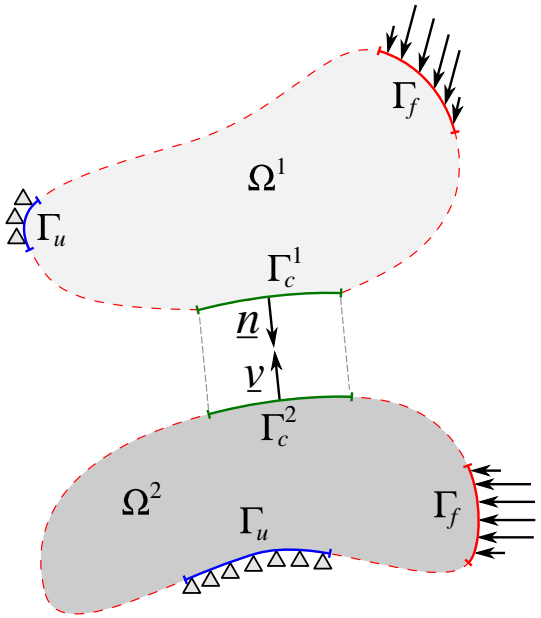
$$\nabla \cdot \underline{\underline{\sigma}} + \underline{f}_v = 0 \quad \text{in } \Omega = \Omega_1 \cup \Omega_2 \qquad + \text{B.C.}$$

- Balance of virtual works 

$$\boxed{\int_{\partial\Omega} \underline{n} \cdot \underline{\underline{\sigma}} \cdot \delta\underline{u}\, d\Gamma} \Rightarrow$$

$$\boxed{\int_{\bar{\Gamma}_c^1} \underline{n} \cdot \underline{\underline{\sigma}} \cdot \delta\underline{\rho}\, d\bar{\Gamma}_c^1 + \int_{\bar{\Gamma}_c^2} \underline{\nu} \cdot \underline{\underline{\sigma}} \cdot \delta\underline{r}\, d\bar{\Gamma}_c^2} =$$

$$= \int_{\bar{\Gamma}_c^1} \underline{n} \cdot \underline{\underline{\sigma}} \cdot \delta(\underline{\rho} - \underline{r})\, d\bar{\Gamma}_c^1 = \int_{\bar{\Gamma}_c^1} \left(\sigma_n \delta g_n + \underset{\sim}{\sigma}_t^T \delta\underset{\sim}{\xi} \right) d\bar{\Gamma}_c^1$$

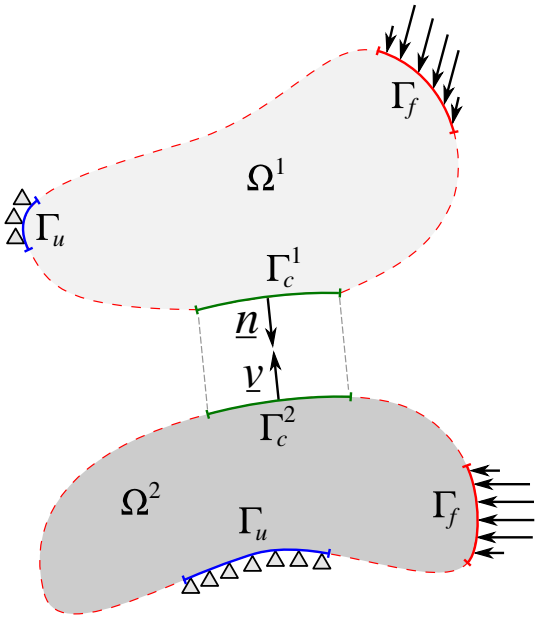


Two solids in contact

From strong to weak form

- Balance of momentum and boundary conditions

$$\nabla \cdot \underline{\underline{\sigma}} + \underline{f}_v = 0 \quad \text{in } \Omega = \Omega_1 \cup \Omega_2 \qquad + \text{B.C.}$$



Two solids in contact

- Balance of virtual works 

$$\boxed{\int_{\partial\Omega} \underline{n} \cdot \underline{\underline{\sigma}} \cdot \delta\underline{u}\, d\Gamma} \Rightarrow$$

$$\boxed{\int_{\bar{\Gamma}_c^1} \underline{n} \cdot \underline{\underline{\sigma}} \cdot \delta\underline{\rho}\, d\bar{\Gamma}_c^1 + \int_{\bar{\Gamma}_c^2} \underline{\nu} \cdot \underline{\underline{\sigma}} \cdot \delta\underline{r}\, d\bar{\Gamma}_c^2} =$$

$$= \int_{\bar{\Gamma}_c^1} \underline{n} \cdot \underline{\underline{\sigma}} \cdot \delta(\underline{\rho} - \underline{r})\, d\bar{\Gamma}_c^1 = \int_{\bar{\Gamma}_c^1} \left(\sigma_n \delta g_n + \underset{\sim}{\sigma}_t^T \delta \underset{\sim}{\xi} \right) d\bar{\Gamma}_c^1$$

$$\int_\Omega \underline{\underline{\sigma}} \cdot\cdot \delta\nabla\underline{u}\, d\Omega + \underbrace{\boxed{\int_{\bar{\Gamma}_c^1} \left(\sigma_n \delta g_n + \underset{\sim}{\sigma}_t^T \delta \underset{\sim}{\xi} \right) d\bar{\Gamma}_c^1}}_{\text{Contact term}} = \int_{\Gamma_f} \underline{\sigma}_0 \cdot \delta\underline{u}\, d\Gamma + \int_\Omega \underline{f}_v \cdot \delta\underline{u}\, d\Omega$$

From strong to weak form

- Balance of momentum and boundary conditions

$$\nabla \cdot \underline{\underline{\sigma}} + \underline{f}_v = 0 \quad \text{in } \Omega = \Omega_1 \cup \Omega_2 \qquad + \text{B.C.}$$

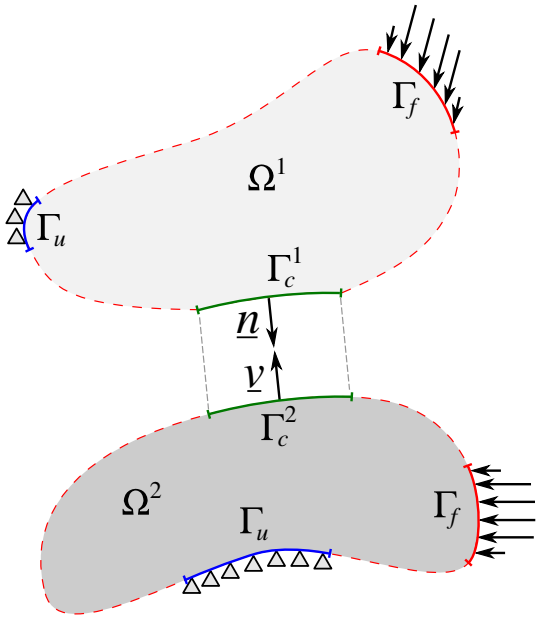
- Balance of virtual works 

$$\underbrace{\int_\Omega \underline{\underline{\sigma}} \cdot\cdot\, \delta\nabla\underline{u}\, d\Omega}_{\text{Change of the internal energy}} + \underbrace{\boxed{\int_{\overline{\Gamma}_c^1} \left(\sigma_n \delta g_n + \underset{\sim}{\sigma}_t^T \delta \underset{\sim}{\xi} \right) d\overline{\Gamma}_c^1}}_{\text{Contact term}} =$$

$$\underbrace{\int_{\Gamma_f} \underline{\sigma}_0 \cdot \delta\underline{u}\, d\Gamma}_{\text{Virtual work of external forces}} + \underbrace{\int_\Omega \underline{f}_v \cdot \delta\underline{u}\, d\Omega}_{\text{Virtual work of volume forces}}$$

- Functional space

$\underline{u} \in \mathbb{H}^1(\Omega)$ Sobolev space of the first order and $\underline{u}$ satisfy boundary and **contact conditions**.



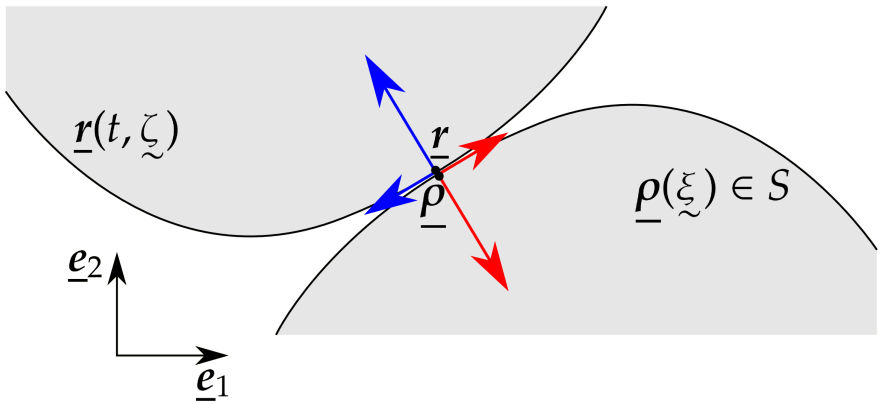
Two solids in contact

Towards variational inequality

- Contact term

$$\int_{\overline{\Gamma}_c^1} \left(\sigma_n \delta g_n + \underset{\sim}{\sigma}_t^T \delta \underset{\sim}{\xi} \right) d\overline{\Gamma}_c^1$$

$$\int_{\overline{\Gamma}_c^1} \sigma_n \delta g_n \, d\overline{\Gamma}_c^1 \le 0$$



Contact configuration $\sigma_n \delta g_n = 0, \ \sigma_n \le 0$

$$\int_\Omega \underline{\underline{\sigma}} \cdot\cdot \delta \nabla \underline{u} \, d\Omega + \int_{\overline{\Gamma}_c^1} \underset{\sim}{\sigma}_t^T \delta \underset{\sim}{\xi} \, d\overline{\Gamma}_c^1 \ge \int_{\Gamma_f} \underline{\sigma}_0 \cdot \delta \underline{u} \, d\Gamma + \int_\Omega \underline{f}_v \cdot \delta \underline{u} \, d\Omega,$$

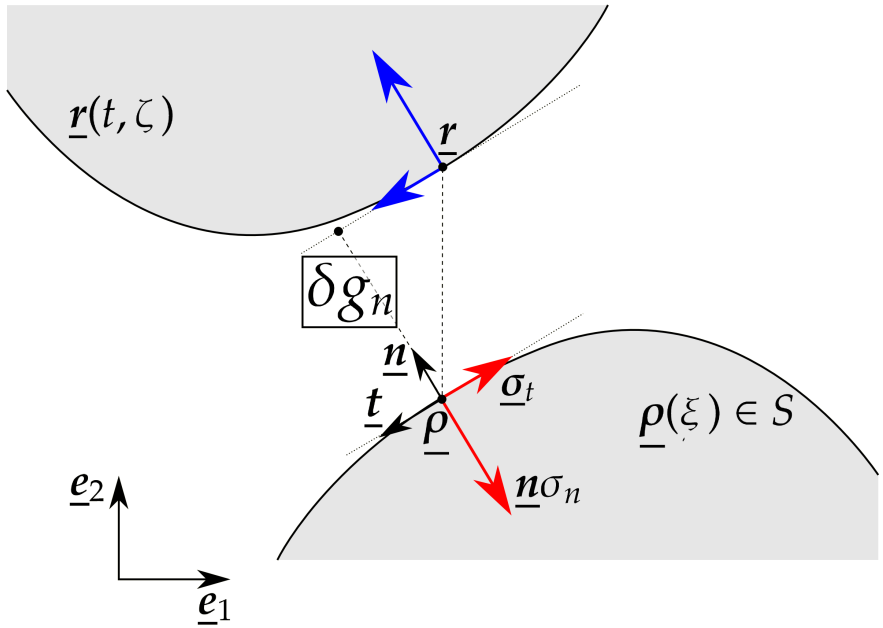
$$\mathbb{K} = \left\{ \delta \underline{u} \in \mathbb{H}^1(\Omega) \ \middle| \ \delta \underline{u} = 0 \ \text{ on } \Gamma_u, \ g_n(\underline{u} + \delta \underline{u}) \ge 0 \text{ on } \Gamma_c \right\}$$

Towards variational inequality

- Contact term

$$\int\limits_{\overline{\Gamma}_c^1} \left(\sigma_n \delta g_n + \underset{\sim}{\sigma}_t^T \delta \underset{\sim}{\xi} \right) d\overline{\Gamma}_c^1$$

$$\int\limits_{\overline{\Gamma}_c^1} \sigma_n \delta g_n \, d\overline{\Gamma}_c^1 \le 0$$



Virtual change of the configuration

$$\int\limits_{\Omega} \underline{\underline{\sigma}} \cdot\cdot \delta \nabla \underline{u} \, d\Omega + \int\limits_{\overline{\Gamma}_c^1} \underset{\sim}{\sigma}_t^T \delta \underset{\sim}{\xi} \, d\overline{\Gamma}_c^1 \ge \int\limits_{\Gamma_f} \underline{\sigma}_0 \cdot \delta \underline{u} \, d\Gamma + \int\limits_{\Omega} \underline{f}_v \cdot \delta \underline{u} \, d\Omega,$$

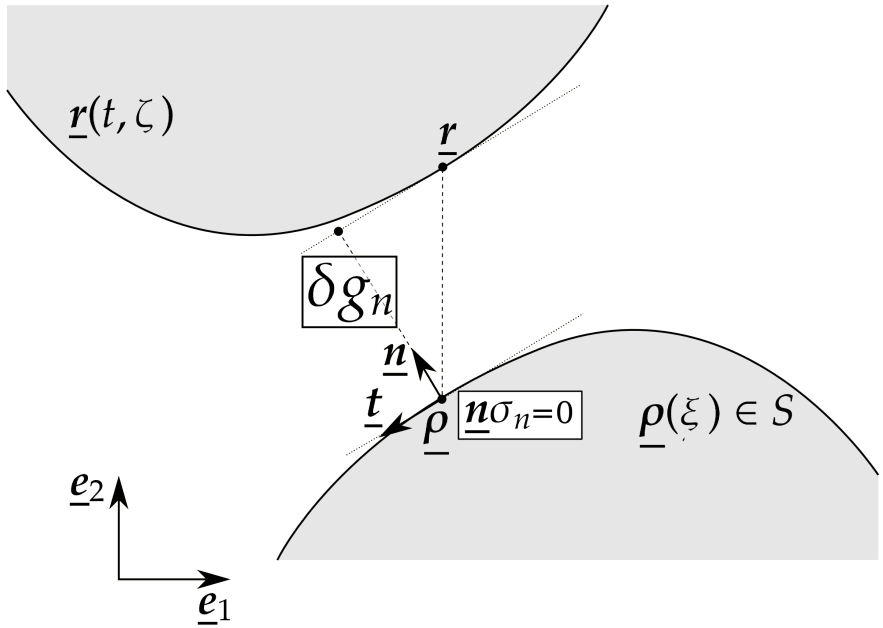
$$\mathbb{K} = \left\{ \delta \underline{u} \in \mathbb{H}^1(\Omega) \;\middle|\; \delta \underline{u} = 0 \;\text{ on } \Gamma_u,\; g_n(\underline{u} + \delta \underline{u}) \ge 0 \text{ on } \Gamma_c \right\}$$

Towards variational inequality

- Contact term

$$\int\limits_{\overline{\Gamma}_c^1} \left(\sigma_n \delta g_n + \underset{\sim}{\sigma}_t^T \delta \underset{\sim}{\xi} \right) d\overline{\Gamma}_c^1$$

$$\int\limits_{\overline{\Gamma}_c^1} \sigma_n \delta g_n \, d\overline{\Gamma}_c^1 \le 0$$



Normal term in separation $\delta g_n > 0$

$$\int\limits_{\Omega} \underline{\underline{\sigma}} \cdot\cdot \delta \nabla \underline{u} \, d\Omega + \int\limits_{\overline{\Gamma}_c^1} \underset{\sim}{\sigma}_t^T \delta \underset{\sim}{\xi} \, d\overline{\Gamma}_c^1 \ge \int\limits_{\Gamma_f} \underline{\sigma}_0 \cdot \delta \underline{u} \, d\Gamma + \int\limits_{\Omega} \underline{f}_v \cdot \delta \underline{u} \, d\Omega,$$

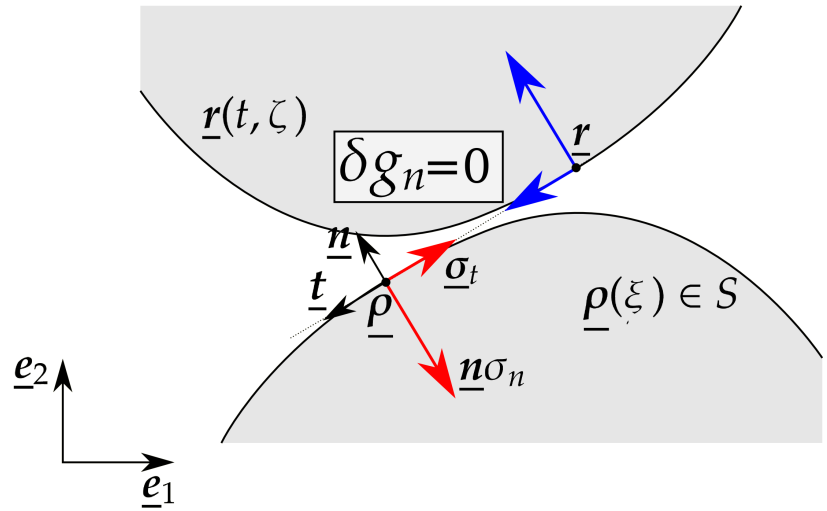
$$\mathbb{K} = \left\{ \delta \underline{u} \in \mathbb{H}^1(\Omega) \;\middle|\; \delta \underline{u} = 0 \;\text{ on } \Gamma_u,\; g_n(\underline{u} + \delta \underline{u}) \ge 0 \text{ on } \Gamma_c \right\}$$

Towards variational inequality

- Contact term

$$\int_{\bar{\Gamma}_c^1} \left(\sigma_n \delta g_n + \underset{\sim}{\sigma}_t^T \delta \underset{\sim}{\xi} \right) d\bar{\Gamma}_c^1$$

$$\int_{\bar{\Gamma}_c^1} \sigma_n \delta g_n \, d\bar{\Gamma}_c^1 \le 0$$



Normal term in sliding $\delta g_n = 0$

$$\int_\Omega \underline{\underline{\sigma}} \cdot\cdot \delta \nabla \underline{u} \, d\Omega + \int_{\bar{\Gamma}_c^1} \underset{\sim}{\sigma}_t^T \delta \underset{\sim}{\xi} \, d\bar{\Gamma}_c^1 \ge \int_{\Gamma_f} \underline{\sigma}_0 \cdot \delta \underline{u} \, d\Gamma + \int_\Omega \underline{f}_v \cdot \delta \underline{u} \, d\Omega,$$

$$\mathbb{K} = \left\{ \delta \underline{u} \in \mathbb{H}^1(\Omega) \;\middle|\; \delta \underline{u} = 0 \;\text{ on } \Gamma_u,\; g_n(\underline{u} + \delta \underline{u}) \ge 0 \text{ on } \Gamma_c \right\}$$

Back to variational equality (unconstrained)

- Constrained minimization problem

$$\int_{\Omega} \underline{\underline{\sigma}} \cdot\cdot \delta\nabla\underline{u}\, d\Omega + \int_{\bar{\Gamma}_c^1} \underset{\sim}{\sigma}_t^T \delta\underset{\sim}{\xi}\, d\bar{\Gamma}_c^1 \;\geq\; \int_{\Gamma_f} \underline{\sigma}_0 \cdot \delta\underline{u}\, d\Gamma + \int_{\Omega} \underline{f}_v \cdot \delta\underline{u}\, d\Omega,$$

$$\mathbb{K} = \left\{ \delta\underline{u} \in \mathbb{H}^1(\Omega) \;\middle|\; \delta\underline{u} = 0 \;\text{ on } \Gamma_u,\; g_n(\underline{u} + \delta\underline{u}) \geq 0 \text{ on } \Gamma_c \right\}$$

- Use optimization theory to convert to

$$\int_{\Omega} \underline{\underline{\sigma}} \cdot\cdot \delta\nabla\underline{u}\, d\Omega + \int_{\Gamma_c^1} \underbrace{\boxed{C(\sigma_n, \sigma_t, g_n, \underset{\sim}{\xi}, \delta\underline{u})}}_{\text{Contact term}^*} d\Gamma_c^1 \;=\; \int_{\Gamma_f} \underline{\sigma}_0 \cdot \delta\underline{u}\, d\Gamma + \int_{\Omega} \underline{f}_v \cdot \delta\underline{u}\, d\Omega,$$

Unconstrained functional space $\mathbb{K} = \left\{ \delta\underline{u} \in \mathbb{H}^1(\Omega) \;\middle|\; \delta\underline{u} = 0 \;\text{ on } \Gamma_u \right\}$

Contact term* is defined on the *potential contact zone* Γ_c^1.

Optimization methods: recall

Functional to be minimized $F(\mathbf{x})$ under constraint $g(\mathbf{x}) \geq 0$

- Penalty method
- Lagrange multipliers method
- Augmented Lagrangian method

Optimization methods: recall

Functional to be minimized $F(\mathbf{x})$ under constraint $g(\mathbf{x}) \geq 0$

- **Penalty method**
  - New functional

$$F_p(\mathbf{x}) = F(\mathbf{x}) + \boxed{\epsilon \langle -g(\mathbf{x}) \rangle^2} = F(\mathbf{x}) + \begin{cases} 0, & \text{if } g(\mathbf{x}) \geq 0 \quad \textit{non-contact} \\ \epsilon g^2(\mathbf{x}), & \text{if } g(\mathbf{x}) < 0 \quad \textit{contact} \end{cases}$$

    where ϵ is the penalty parameter.
  - Stationary point must satisfy

$$\nabla F_p(\mathbf{x}) = \nabla F(\mathbf{x}) + 2\epsilon \langle -g(\mathbf{x}) \rangle \nabla g(\mathbf{x}) = 0$$

  - Solution **tends** to the precise solution as $\epsilon \to \infty$
- Lagrange multipliers method
- Augmented Lagrangian method

Macaulay brackets $\langle x \rangle = \begin{cases} x, & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases}$

Optimization methods: recall

Functional to be minimized $F(\mathbf{x})$ under constraint $g(\mathbf{x}) \geq 0$

- Penalty method $F_p(\mathbf{x}) = F(\mathbf{x}) + \boxed{\epsilon \langle -g(\mathbf{x}) \rangle^2}$
- **Lagrange multipliers method**
  - New functional called **Lagrangian**

$$\mathcal{L}(\mathbf{x}, \lambda) = F(\mathbf{x}) + \boxed{\lambda g(\mathbf{x})}$$

  - Saddle point problem

$$\min_x \max_\lambda \{\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda})\} \longrightarrow \mathbf{x}^* \longleftarrow \min_{g(\mathbf{x}) \geq 0} \{F(\mathbf{x})\}$$

  - Stationary point

$$\nabla_{\mathbf{x},\lambda} \mathcal{L} = \begin{bmatrix} \nabla_{\mathbf{x}} F(\mathbf{x}) + \lambda \nabla_x g(\mathbf{x}) \\ g(\mathbf{x}) \end{bmatrix} = 0 \quad \text{need to verify} \quad \lambda \leq 0$$

- Augmented Lagrangian method

Macaulay brackets $\langle x \rangle = \begin{cases} x, & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases}$

Optimization methods: recall

Functional to be minimized $F(\mathbf{x})$ under constraint $g(\mathbf{x}) \geq 0$

- Penalty method $F_p(\mathbf{x}) = F(\mathbf{x}) + \boxed{\epsilon \langle -g(\mathbf{x}) \rangle^2}$
- Lagrange multipliers method $\mathcal{L}(\mathbf{x}, \lambda) = F(\mathbf{x}) + \boxed{\lambda g(\mathbf{x})}$
- **Augmented Lagrangian method**

  [Hestnes 1969], [Powell 1969], [Glowinski & Le Tallec 1989], [Alart & Curnier 1991], [Simo & Laursen 1992]

  - New functional, augmented Lagrangian

$$\mathcal{L}_a(\mathbf{x}, \lambda) = F(\mathbf{x}) + \begin{cases} \boxed{\lambda g(\mathbf{x})} + \boxed{\epsilon g^2(\mathbf{x})}, & \text{if } \lambda + 2\epsilon g(\mathbf{x}) \geq 0, \ \textit{contact} \\ -\frac{1}{4\epsilon}\lambda^2, & \text{if } \lambda + 2\epsilon g(\mathbf{x}) < 0, \ \textit{non-contact} \end{cases}$$

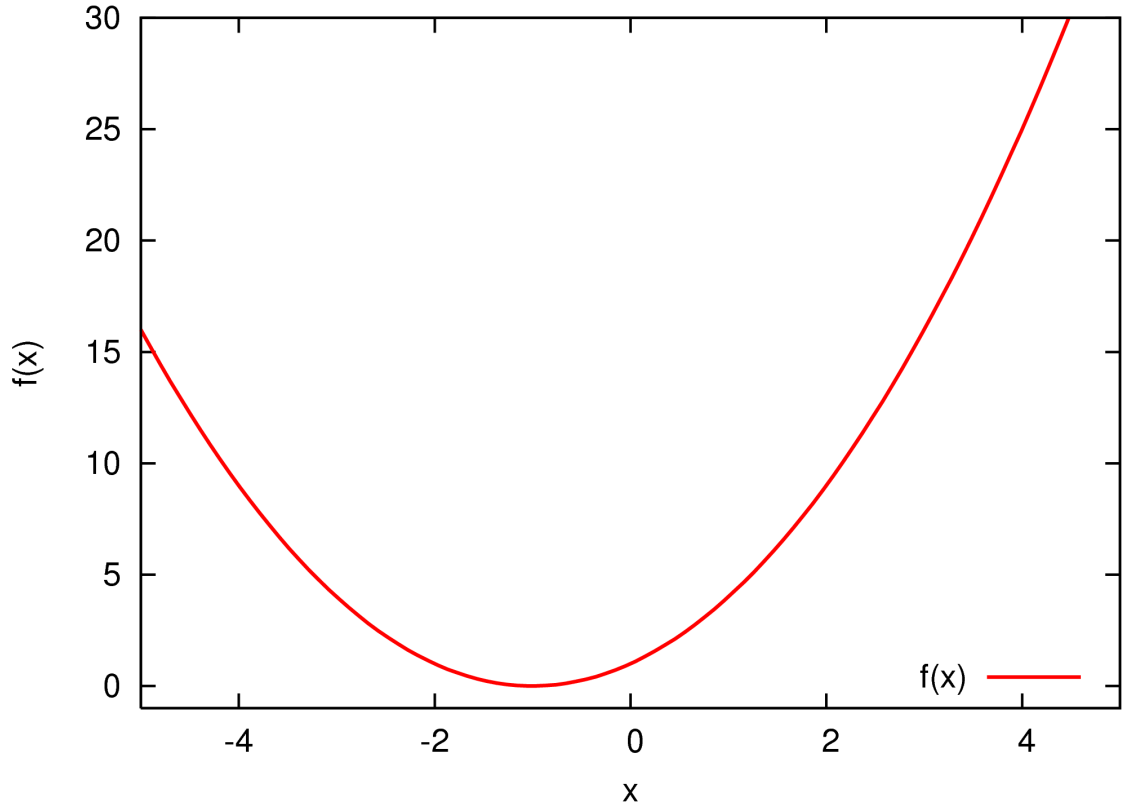
  - Stationary point

$$\nabla_{\mathbf{x},\lambda}\mathcal{L}_a = \begin{cases} \begin{bmatrix} \nabla_{\mathbf{x}} F(\mathbf{x}) + \lambda \nabla_x g(\mathbf{x}) + 2\epsilon g(\mathbf{x}) \nabla g(\mathbf{x}) \\ g(\mathbf{x}) \end{bmatrix} = 0, & \textit{if contact} \\ \begin{bmatrix} \nabla_{\mathbf{x}} F(\mathbf{x}) \\ -\frac{\lambda}{\epsilon} \end{bmatrix} = 0, & \textit{if non-contact} \end{cases}$$

Macaulay brackets $\langle x \rangle = \begin{cases} x, & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases}$


Uzawa algorithm

Optimization methods: example

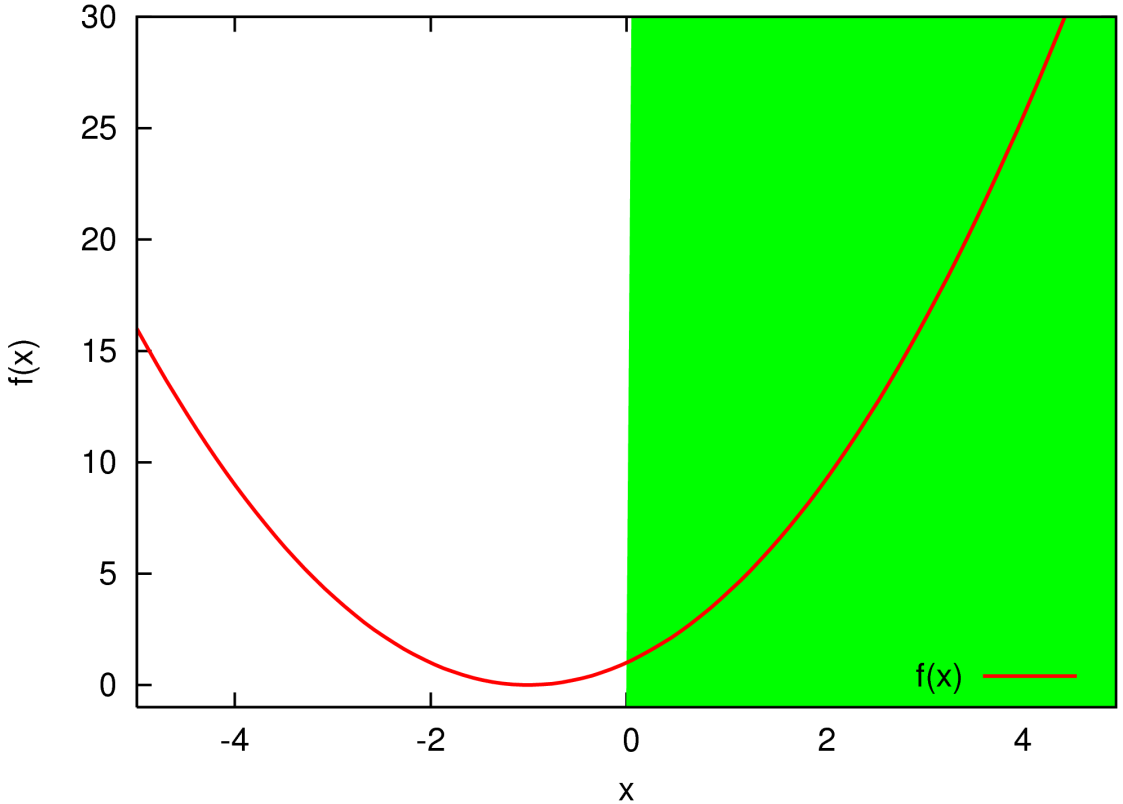


Functional : $f(x) = x^2 + 2x + 1$

Constrain : $g(x) = x \geq 0$

Solution : $x^* = 0$

Optimization methods: example



Functional : $f(x) = x^2 + 2x + 1$

Constrain : $g(x) = x \geq 0$

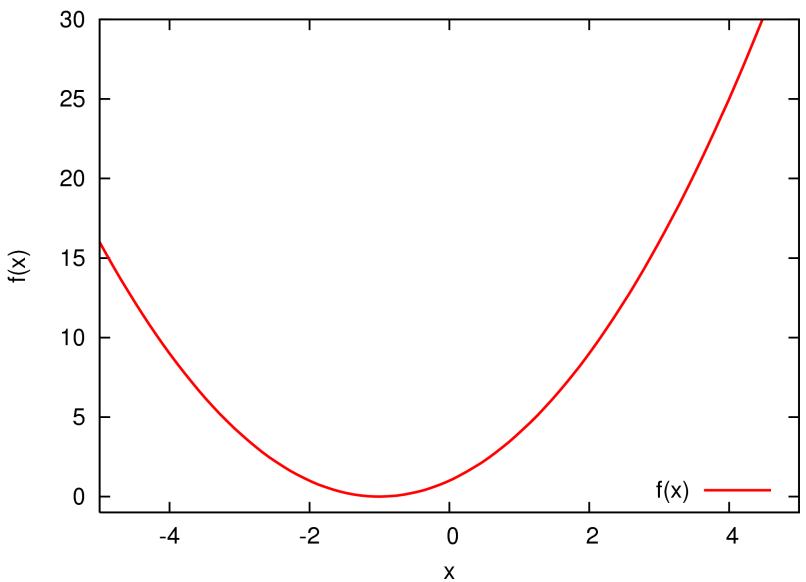
Solution : $x^* = 0$

Penalty method: example

$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

- **Penalty method**

$$F_p(x) = F(x) + \boxed{\epsilon \langle -g(x) \rangle^2}$$

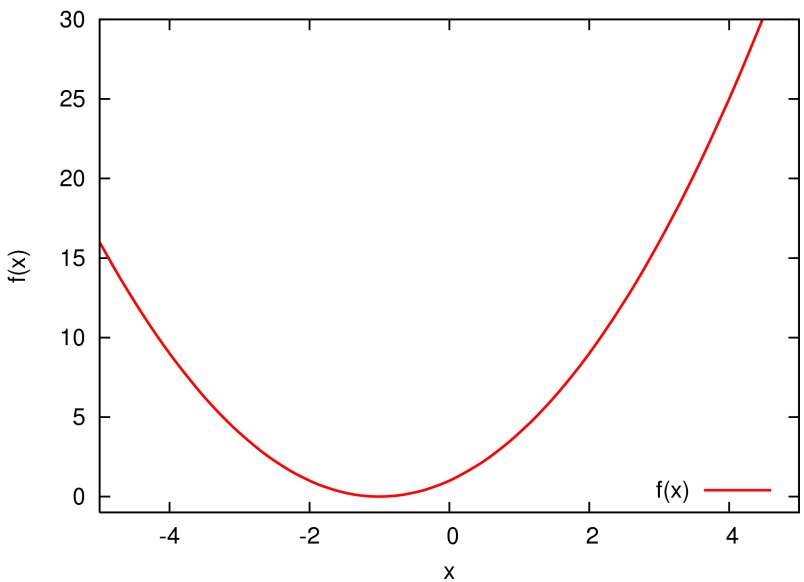


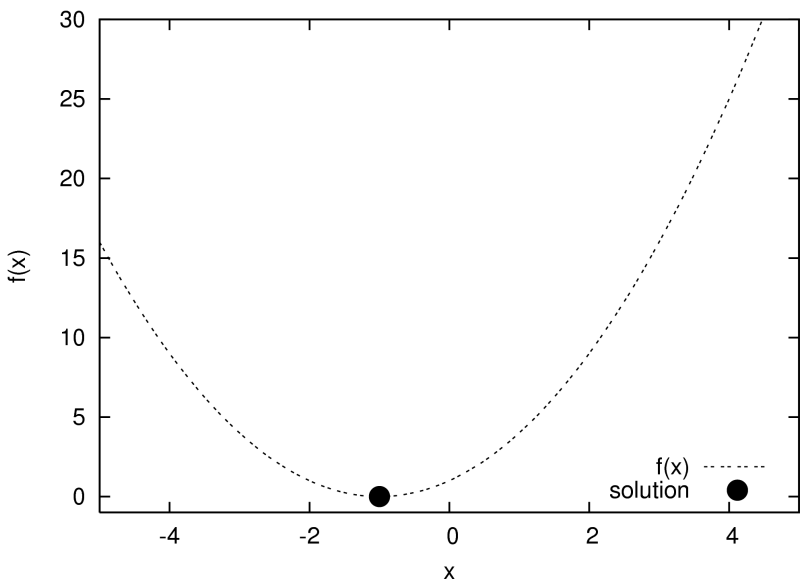
Penalty method: example

$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

- **Penalty method**

$$F_p(x) = F(x) + \boxed{\epsilon \langle -g(x) \rangle^2}$$





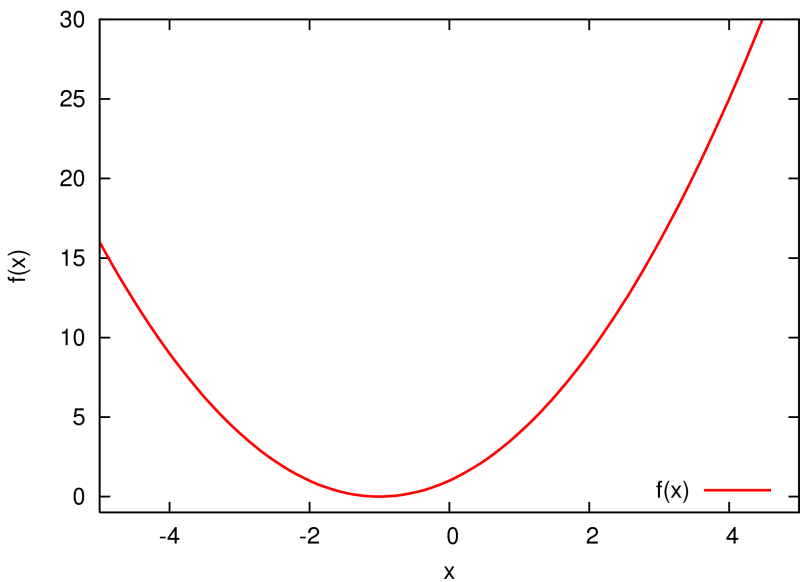
$\epsilon = 0$

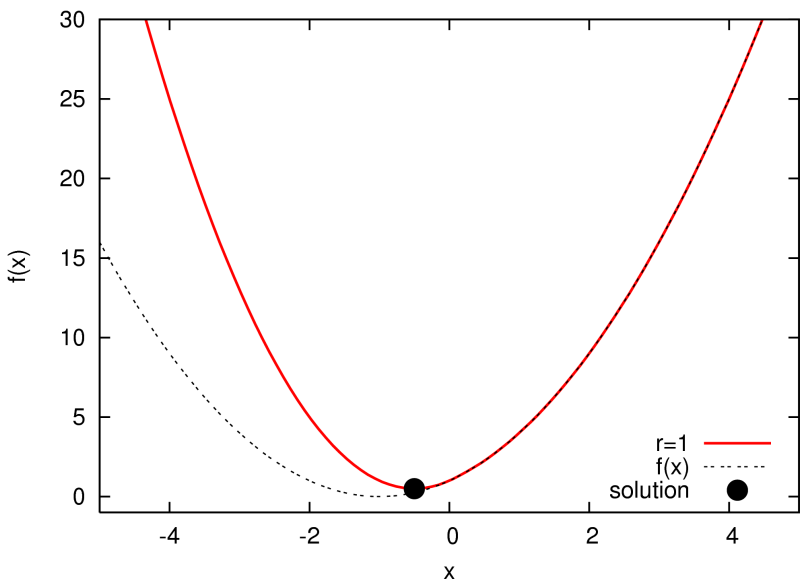
Penalty method: example

$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

- **Penalty method**

$$F_p(x) = F(x) + \boxed{\epsilon \langle -g(x) \rangle^2}$$





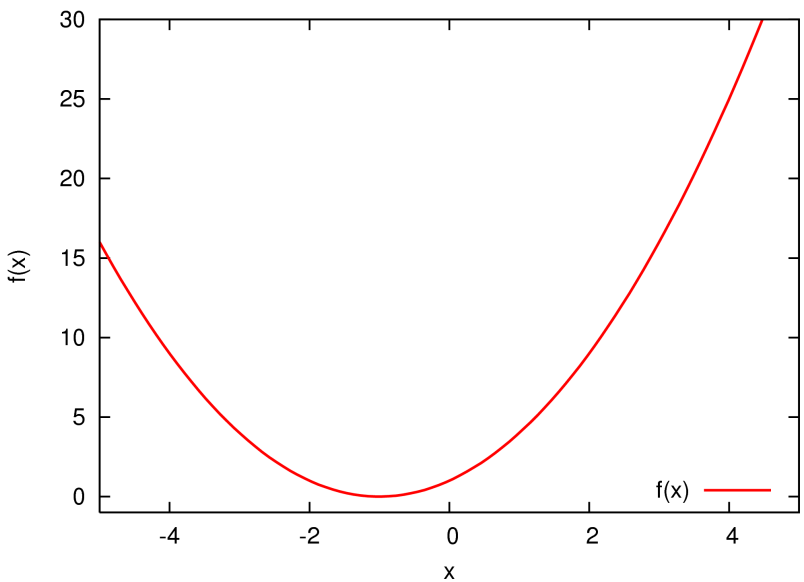
$\epsilon = 1$

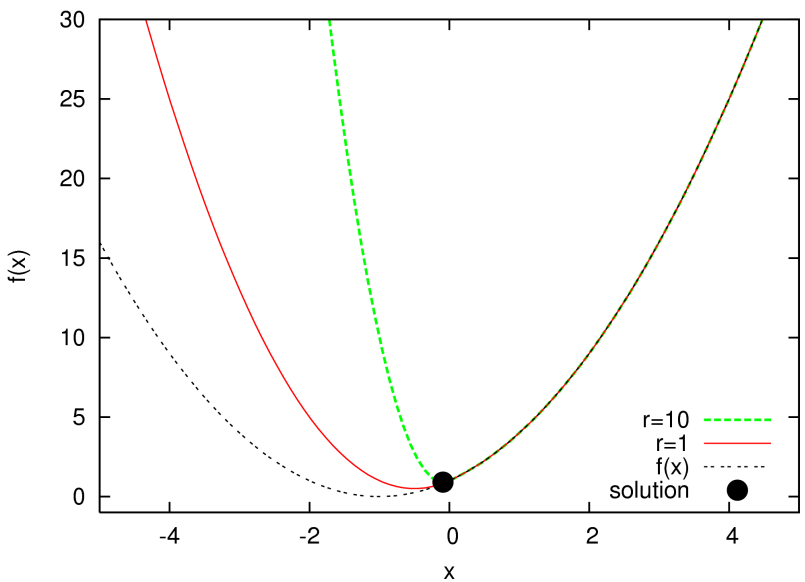
Penalty method: example

$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

- **Penalty method**

$$F_p(x) = F(x) + \boxed{\epsilon \langle -g(x) \rangle^2}$$





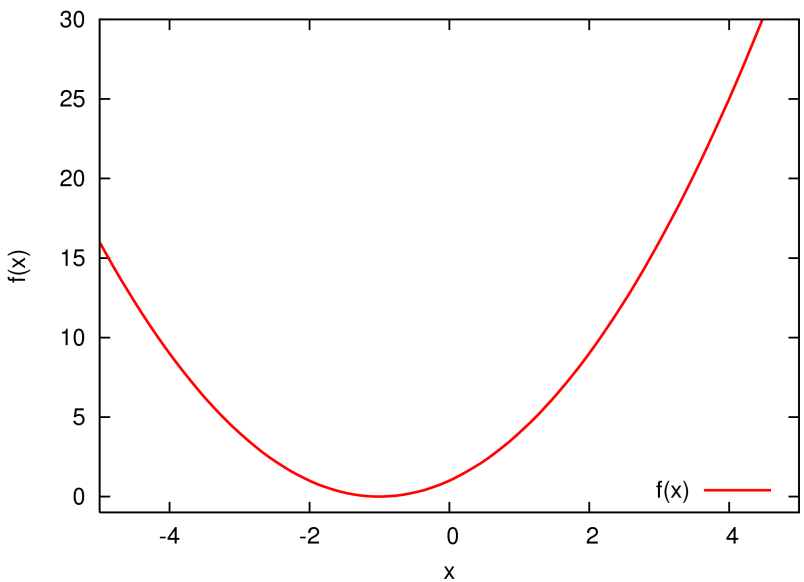
$\epsilon = 10$

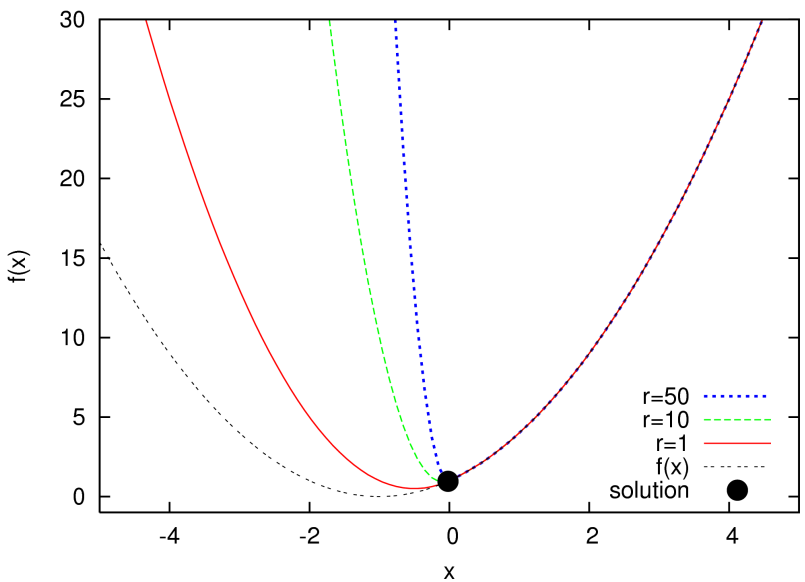
Penalty method: example

$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

- **Penalty method**

$$F_p(x) = F(x) + \boxed{\epsilon \langle -g(x) \rangle^2}$$





$\epsilon = 50$

Penalty method: example

$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

- **Penalty method**

$$F_p(x) = F(x) + \boxed{\epsilon \langle -g(x) \rangle^2}$$

Advantages ☺

- simple physical interpretation
- simple implementation
- no additional degrees of freedom
- "mathematically" smooth functional

Drawbacks ☹

- practically non-smooth functional
- solution is not exact:
  - too small penalty → large penetration
  - too large penalty → ill-conditioning of the tangent matrix
- user has to choose penalty ϵ properly or automatically and/or adapt during convergence

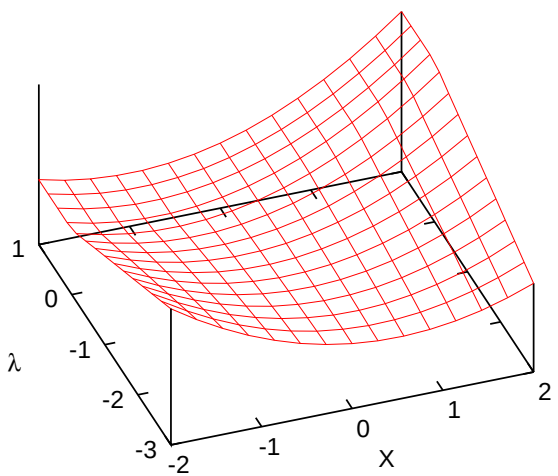
Lagrange multipliers method: example

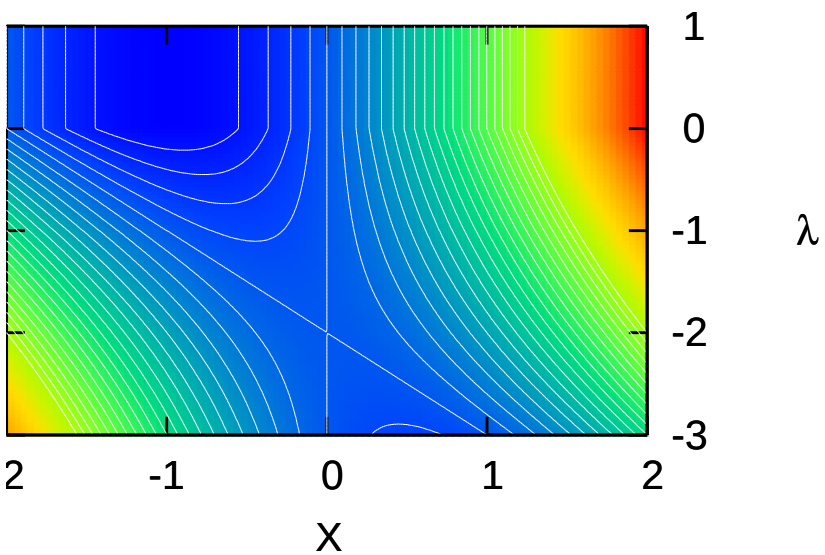
$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

- **Lagrange multipliers method**

$\mathcal{L}(x, \lambda) = F(x) + \boxed{\lambda g(x)} \rightarrow$ Saddle point $\rightarrow \min_x \max_\lambda \mathcal{L}(x, \lambda)$

Need to check that $\lambda \leq 0$





Lagrange multipliers method: example

$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

- **Lagrange multipliers method**

  $\mathcal{L}(x, \lambda) = F(x) + \boxed{\lambda g(x)} \rightarrow$ Saddle point $\rightarrow \min_x \max_\lambda \mathcal{L}(x, \lambda)$

  Need to check that $\lambda \leq 0$

Advantages ☺

- exact solution
- no adjustable parameters

Drawbacks ☹

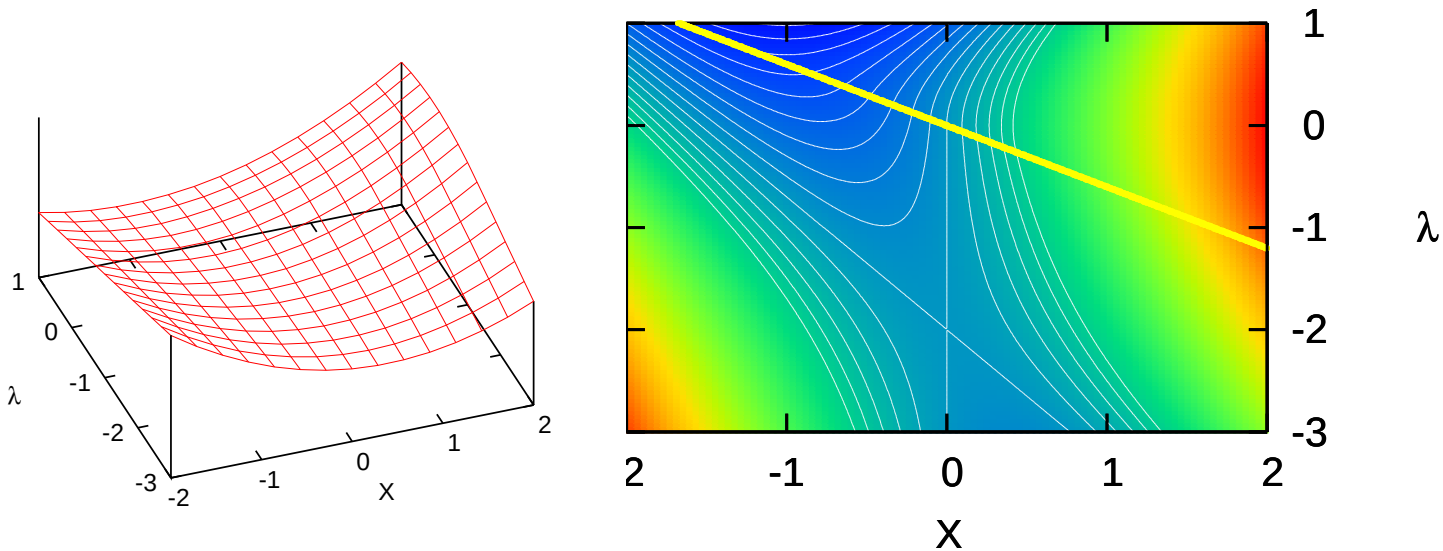
- Lagrangian is not smooth
- additional degrees of freedom
- not fully unconstrained: $\lambda \leq 0$

Augmented Lagrangian method: example

$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

- **Augmented Lagrangian method**

$$\mathcal{L}_a(\mathbf{x}, \lambda) = F(\mathbf{x}) + \begin{cases} \boxed{\lambda g(\mathbf{x})} + \boxed{\epsilon g^2(\mathbf{x})}, & \text{if } \lambda + 2\epsilon g(\mathbf{x}) \geq 0, \ \textit{contact} \\ -\frac{1}{4\epsilon}\lambda^2, & \text{if } \lambda + 2\epsilon g(\mathbf{x}) < 0, \ \textit{non-contact} \end{cases}$$



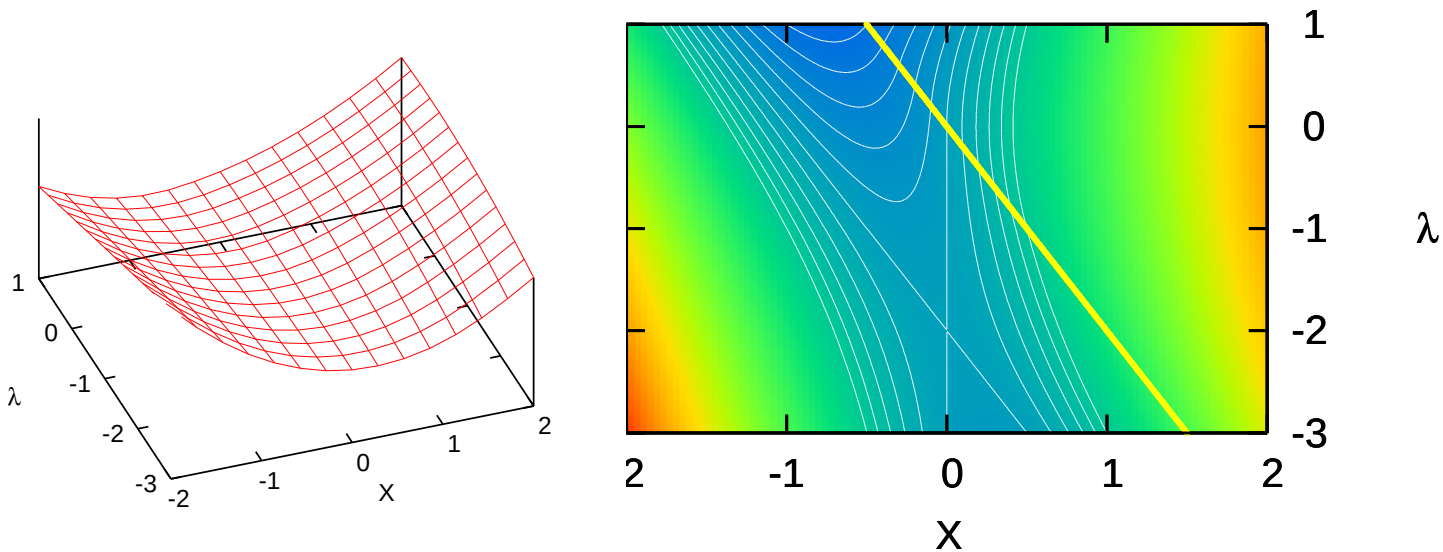
Yellow line separates contact and non-contact regions

Augmented Lagrangian method: example

$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

- **Augmented Lagrangian method**

$$\mathcal{L}_a(\mathbf{x}, \lambda) = F(\mathbf{x}) + \begin{cases} \boxed{\lambda g(\mathbf{x})} + \boxed{\epsilon g^2(\mathbf{x})}, & \text{if } \lambda + 2\epsilon g(\mathbf{x}) \geq 0, \ \textit{contact} \\ -\frac{1}{4\epsilon}\lambda^2, & \text{if } \lambda + 2\epsilon g(\mathbf{x}) < 0, \ \textit{non-contact} \end{cases}$$



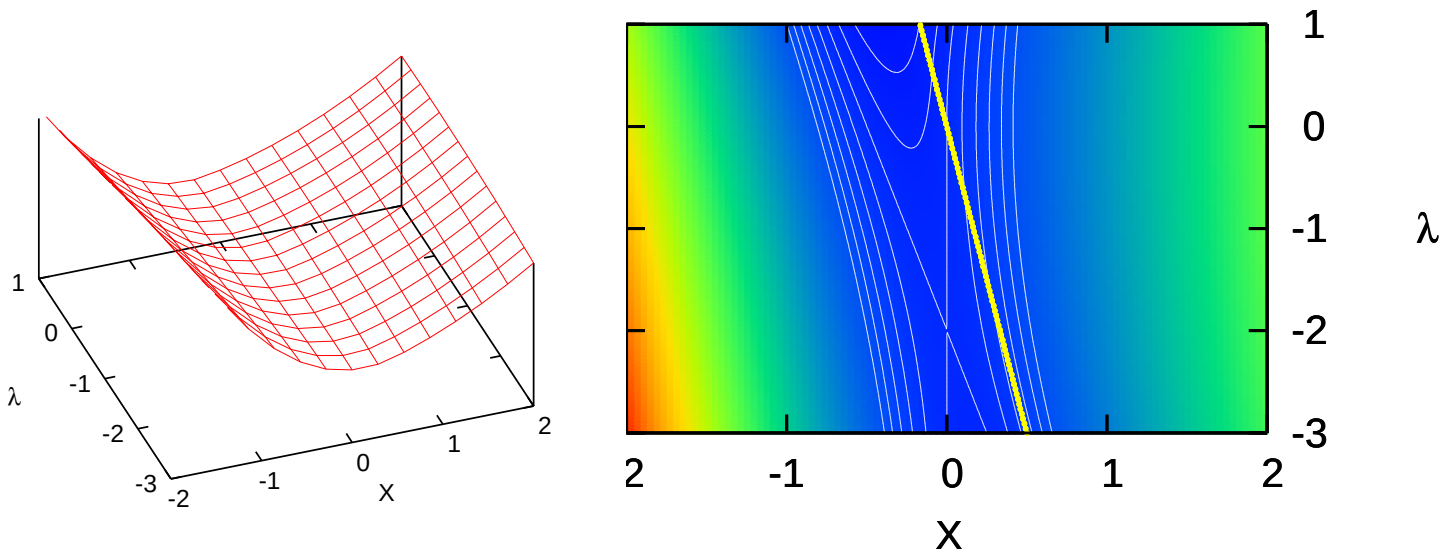
Yellow line separates contact and non-contact regions

Augmented Lagrangian method: example

$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

- **Augmented Lagrangian method**

$$\mathcal{L}_a(\mathbf{x}, \lambda) = F(\mathbf{x}) + \begin{cases} \boxed{\lambda g(\mathbf{x})} + \boxed{\epsilon g^2(\mathbf{x})}, & \text{if } \lambda + 2\epsilon g(\mathbf{x}) \geq 0, \ \textit{contact} \\ -\frac{1}{4\epsilon}\lambda^2, & \text{if } \lambda + 2\epsilon g(\mathbf{x}) < 0, \ \textit{non-contact} \end{cases}$$



Yellow line separates contact and non-contact regions

Augmented Lagrangian method: example

$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

- **Augmented Lagrangian method**

$$\mathcal{L}_a(\mathbf{x}, \lambda) = F(\mathbf{x}) + \begin{cases} \boxed{\lambda g(\mathbf{x})} + \boxed{\epsilon g^2(\mathbf{x})}, & \text{if } \lambda + 2\epsilon g(\mathbf{x}) \geq 0, \ \textit{contact} \\ -\frac{1}{4\epsilon}\lambda^2, & \text{if } \lambda + 2\epsilon g(\mathbf{x}) < 0, \ \textit{non-contact} \end{cases}$$

Advantages ☺

- exact solution
- smooth functional (!)
- fully unconstrained

Drawbacks ☹

- additional degrees of freedom
- quite sensitive to parameter ϵ
- need to adjust ϵ during convergence

Application to contact problems: weak form

$$\int_\Omega \underline{\underline{\sigma}} \cdot\cdot \delta\nabla\underline{\boldsymbol{u}}\, d\Omega + \int_{\Gamma_c^1} \underbrace{\boxed{C}}_{\text{Contact term}} d\Gamma_c^1 = \int_{\Gamma_f} \underline{\boldsymbol{\sigma}}_0 \cdot \delta\underline{\boldsymbol{u}}\, d\Gamma + \int_\Omega \underline{\boldsymbol{f}}_v \cdot \delta\underline{\boldsymbol{u}}\, d\Omega,$$

$$\mathbb{K} = \left\{ \delta\underline{\boldsymbol{u}} \in \mathbb{H}^1(\Omega) \;\middle|\; \delta\underline{\boldsymbol{u}} = 0 \;\text{ on } \Gamma_u \right\}$$

- Penalty method

Pressure: $\sigma_n = \epsilon g_n$, Shear: $\underline{\boldsymbol{\sigma}}_t = \begin{cases} \epsilon \underline{\boldsymbol{g}}_t, & \text{if stick } |\sigma_t| < \mu|\sigma_n| \\ \mu\epsilon g_n \delta\underline{\boldsymbol{g}}_t / |\delta\underline{\boldsymbol{g}}_t|, & \text{if slip } |\sigma_t| = \mu|\sigma_n| \end{cases}$

Contact term

$$C = C(g_n, \underline{\boldsymbol{g}}_t, \delta g_n, \delta\underline{\boldsymbol{g}}_t) = \sigma_n \delta g_n + \underline{\boldsymbol{\sigma}}_t \cdot \delta\underline{\boldsymbol{g}}_t$$

Application to contact problems: weak form

$$\int_\Omega \underline{\underline{\sigma}} \cdot\cdot \delta\nabla\underline{u}\, d\Omega + \int_{\Gamma_c^1} \underbrace{\boxed{C}}_{\text{Contact term}} d\Gamma_c^1 = \int_{\Gamma_f} \underline{\sigma}_0 \cdot \delta\underline{u}\, d\Gamma + \int_\Omega \underline{f}_v \cdot \delta\underline{u}\, d\Omega,$$

$$\mathbb{K} = \left\{ \delta\underline{u} \in \mathbb{H}^1(\Omega) \;\middle|\; \delta\underline{u} = 0 \;\text{ on } \Gamma_u \right\}$$

- Augmented Lagrangian method

Contact term

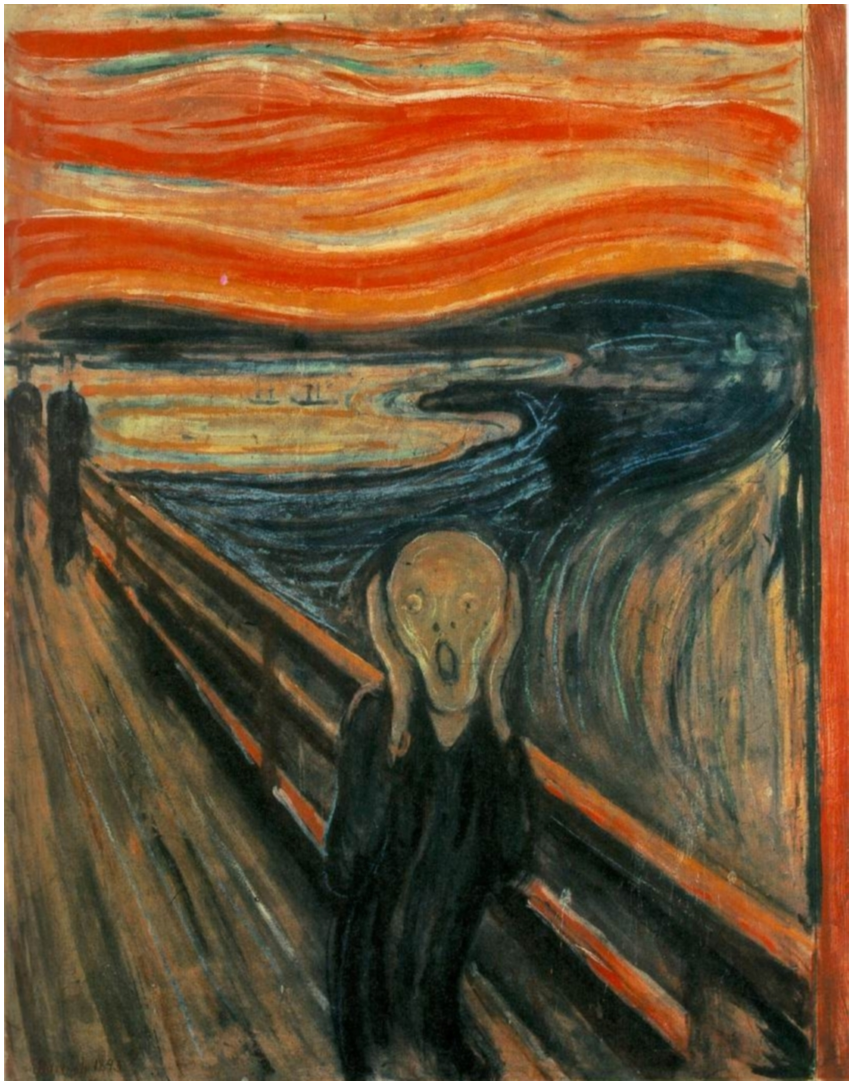
$C = C(g_n, \underline{g}_t, \lambda_n, \underline{\lambda}_t, \delta g_n, \delta\underline{g}_t, \delta\lambda_n, \delta\underline{\lambda}_t)$

$$C = \begin{cases} -\frac{1}{\epsilon}\left(\lambda_n \delta\lambda_n - \underline{\lambda}_t \cdot \delta\underline{\lambda}_t\right), & \text{if non-contact } \lambda_n + \epsilon g_n \geq 0 \\ \hat{\lambda}_n \delta g_n + g_n \delta\lambda_n + \underline{\hat{\lambda}}_t \cdot \delta\underline{g}_t + \underline{g}_t \cdot \delta\underline{\hat{\lambda}}_t, & \text{if stick } |\underline{\hat{\lambda}}_t| \leq \mu|\hat{\sigma}_n| \\ \hat{\lambda}_n \delta g_n + g_n \delta\lambda_n + \mu\hat{\sigma}_n - \mu\hat{\sigma}_n \dfrac{\underline{\hat{\lambda}}_t}{|\underline{\hat{\lambda}}_t|} \cdot \delta\underline{g}_t - \frac{1}{\epsilon}\left(\lambda_t + \mu\hat{\sigma}_n \dfrac{\underline{\hat{\lambda}}_t}{|\underline{\hat{\lambda}}_t|}\right) \cdot \delta\underline{\lambda}_t, & \text{if slip } |\underline{\hat{\lambda}}_t| \geq \mu|\hat{\sigma}_n| \end{cases}$$

where $\hat{\lambda}_n = \lambda_n + \epsilon g_n$ and $\underline{\hat{\lambda}}_t = \underline{\lambda}_t + \epsilon\underline{g}_t$.

Friction . . . . . .

Friction . . . . . .



The scream

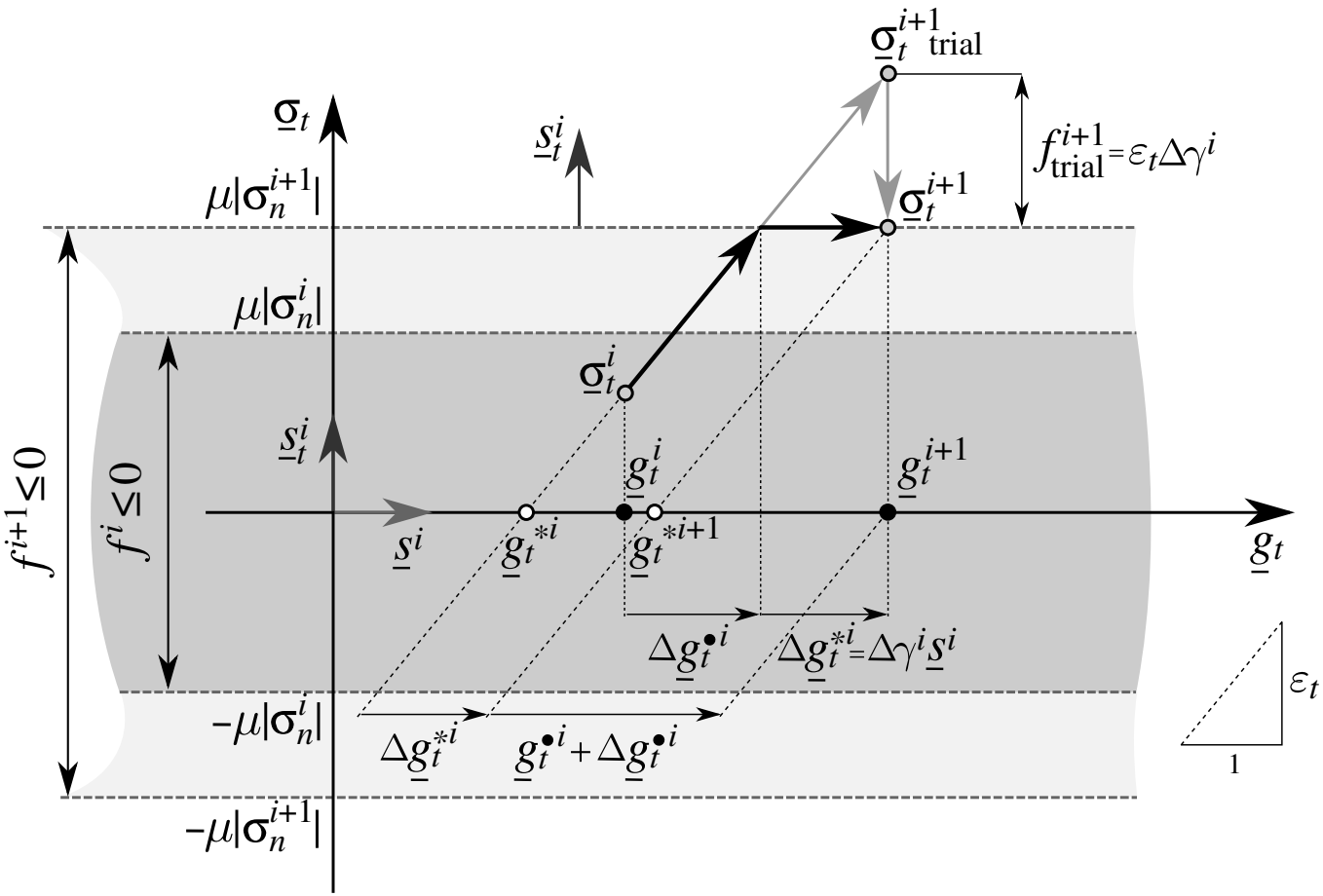
Friction: methods

- Optimization methods: penalty or augmented Lagrangian method
- Return mapping algorithm for penalty
- Analogy with elasto-plastic formulation problem[1]

[1] Curnier "A theory of friction" Int J Solids Struct 20 (1984)

Friction: Return mapping algorithm

- Return mapping algorithm in 2D

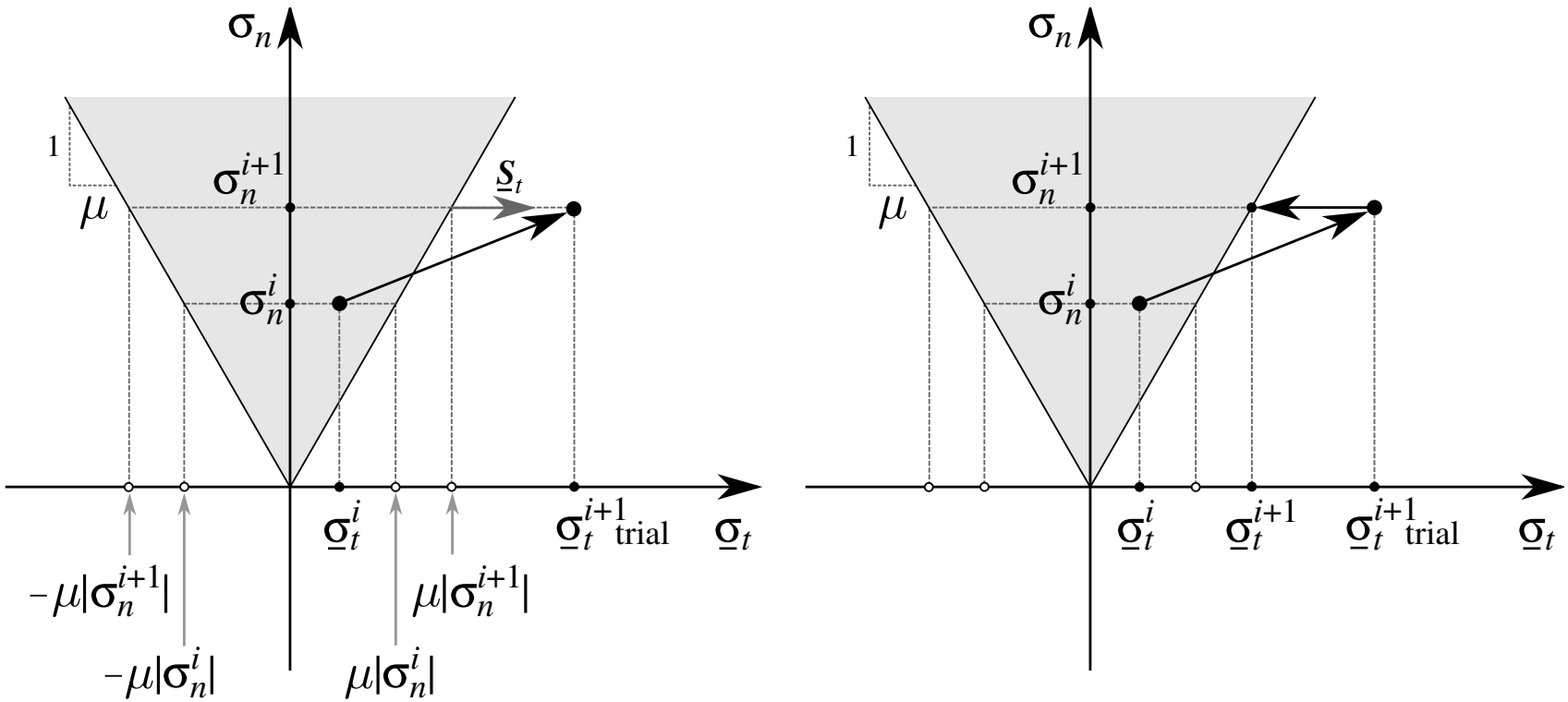


As in plasticity[1]

[1] Simo J.C. and Hughes T.J.. Computational inelasticity. Springer (2006)

Friction: Return mapping algorithm

- Return mapping algorithm in 2D

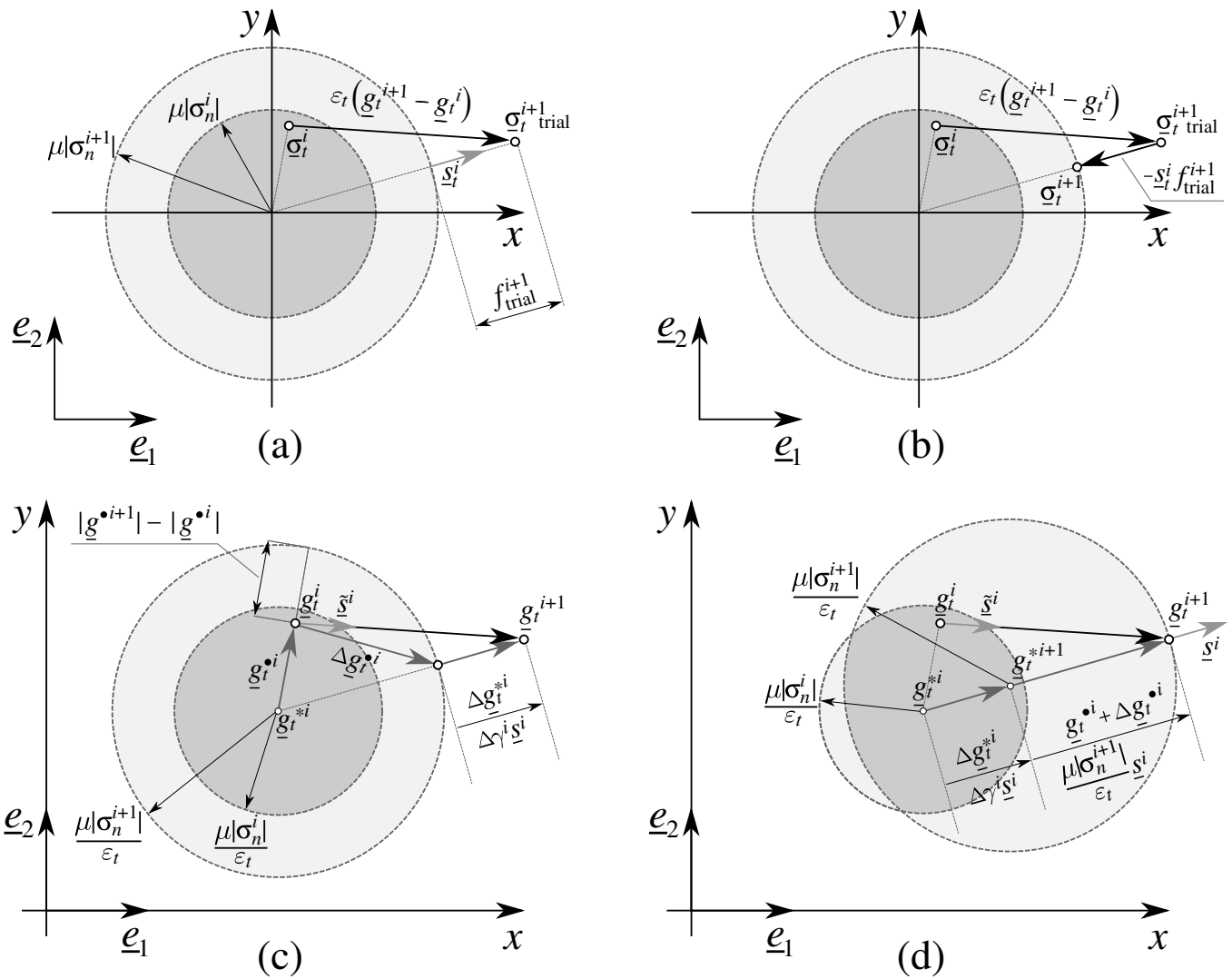


Analogy with non-associated plastic flow[2]

[2] Curnier A. A theory of friction. International Journal of Solids and Structures 20 (1984)

Friction: Return mapping algorithm

- Return mapping algorithm in 3D



A sligthly more messy thing

Application to contact problems: linearization

- Non-linear equation

$$R(\underline{u}, \underline{f}) = 0$$

- Contains $\delta g_n, \delta \underline{g}_t$
- Use Newton-Raphson method
- Initial state at step i

$$R(\underline{u}^i, \underline{f}^i) = 0$$

- Should be also satisfied at step $i+1$

$$R(\underline{u}^{i+1}, \underline{f}^{i+1}) = R(\underline{u}^i + \delta \underline{u}, \underline{f}^{i+1}) = 0$$

- Linearize

$$R(\underline{u}^i + \delta \underline{u}, \underline{f}^{i+1}) = R(\underline{u}^i, \underline{f}^{i+1}) + \frac{\partial R(\underline{u})}{\partial \underline{u}} \delta \underline{u} = 0$$

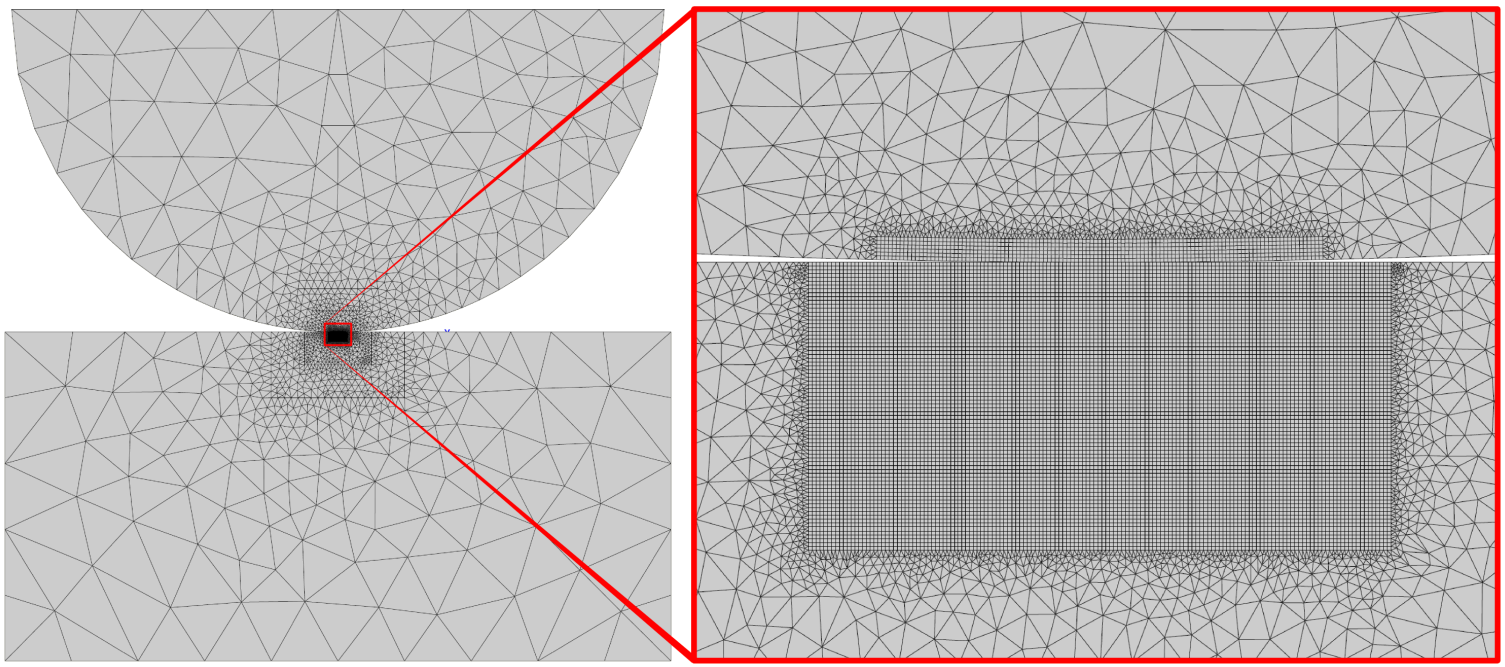
- Finally

$$\delta \underline{u} = - \underbrace{\left[\frac{\partial R(\underline{u})}{\partial \underline{u}} \right]^{-1}}_{\text{contains } \Delta\delta g_n, \Delta\delta \underline{g}_t} R(\underline{u}^i)$$

- NB: Contact problem does not satisfy conditions of Kantorovich theorem on the convergence of Newton's method.

Particularities: mesh and convergence

- Strong **mesh refinement** is required
  - especially at **unknown edges** of contact zones

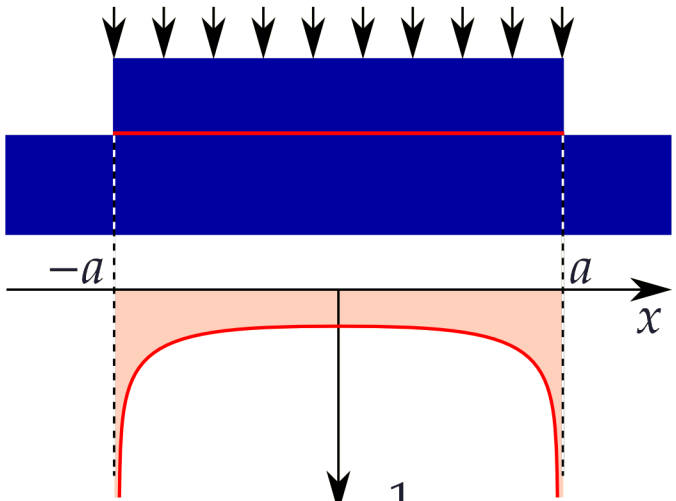


Typical mesh for fretting analysis [L. Sun, H. Proudhon, G. Cailletaud, 2011]

2D ~ 30 000 DoFs, 3D ~ 5 000 000 DoFs

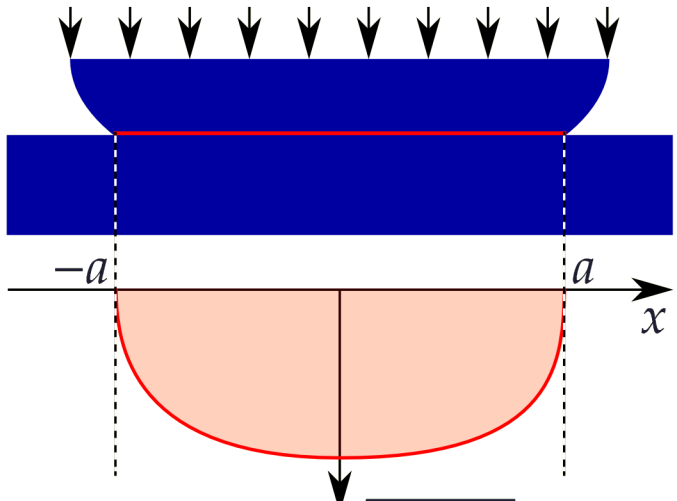
Particularities: mesh and convergence

- Strong **mesh refinement** is required
  - especially at **unknown edges** of contact zones



$$\sigma_n \sim \frac{1}{\sqrt{a^2 - x^2}}$$

$$\sigma_n \xrightarrow[x \to a]{} -\infty \qquad \left|\frac{\partial \sigma_n}{\partial x}\right| \xrightarrow[x \to a]{} \infty$$



$$\sigma_n \sim \sqrt{a^2 - x^2}$$

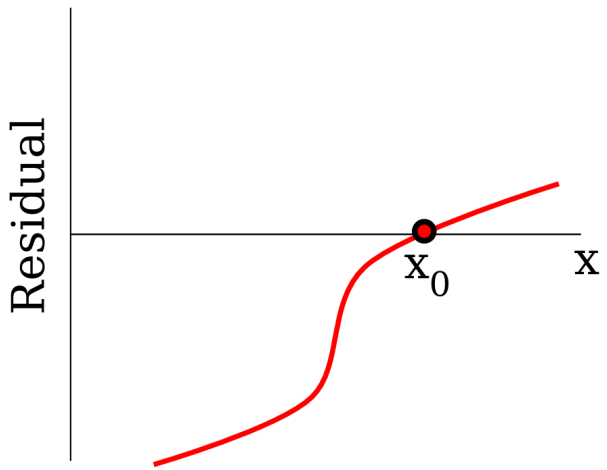
$$\left|\frac{\partial \sigma_n}{\partial x}\right| \xrightarrow[x \to a]{} \infty$$

Infinite contact pressure and/or its derivative

Particularities: mesh and convergence

- Strong **mesh refinement** is required
  - especially at **unknown edges** of contact zones
- **Slow change** of boundary conditions:
  - strong non-linearities of contact / friction problems
  - non-uniqueness of solution for frictional problems

Infinite looping

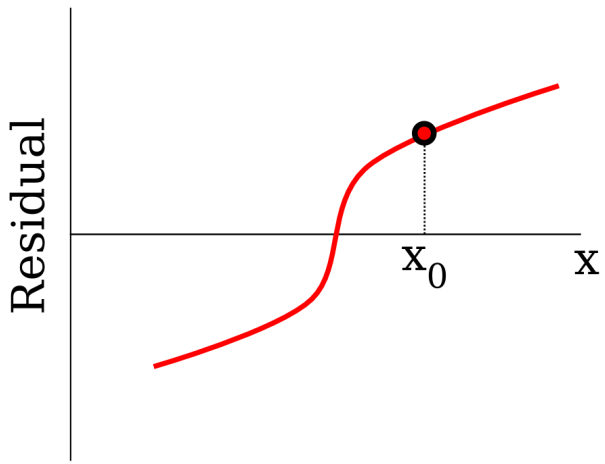


Initial guess $R(x_0, f_0) = 0$

Particularities: mesh and convergence

- Strong **mesh refinement** is required
  - especially at **unknown edges** of contact zones
- **Slow change** of boundary conditions:
  - strong non-linearities of contact / friction problems
  - non-uniqueness of solution for frictional problems

Infinite looping

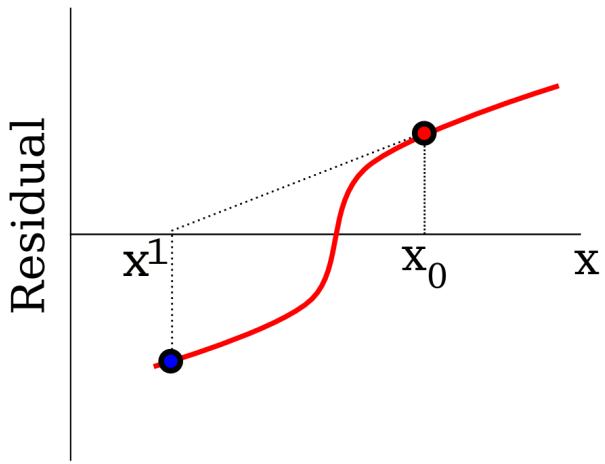


Too rapid change in boundary conditions $R(x_0, f_1) \neq 0$

Particularities: mesh and convergence

- Strong **mesh refinement** is required
  - especially at **unknown edges** of contact zones
- **Slow change** of boundary conditions:
  - strong non-linearities of contact / friction problems
  - non-uniqueness of solution for frictional problems

Infinite looping



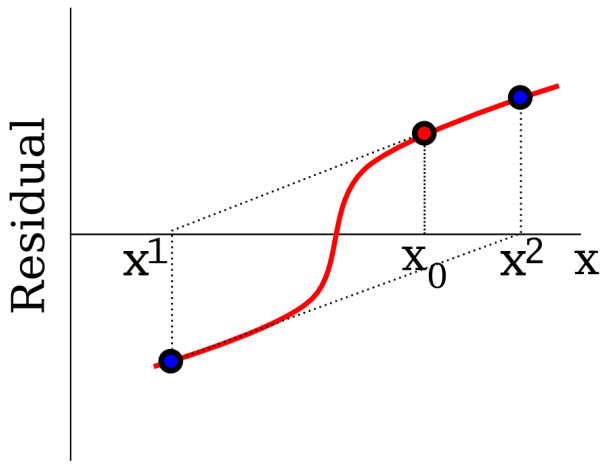
Iterations of Newton-Raphson method

$$R(x_0, f_1) + \left.\frac{\partial R}{\partial x}\right|_{x_0} \delta x = 0 \rightarrow \delta x = -\left.\frac{\partial R}{\partial x}\right|_{x_0}^{-1} R(x_0, f_1) \rightarrow x^1 = x_0 + \delta x$$

Particularities: mesh and convergence

- Strong **mesh refinement** is required
  - especially at **unknown edges** of contact zones
- **Slow change** of boundary conditions:
  - strong non-linearities of contact / friction problems
  - non-uniqueness of solution for frictional problems

Infinite looping



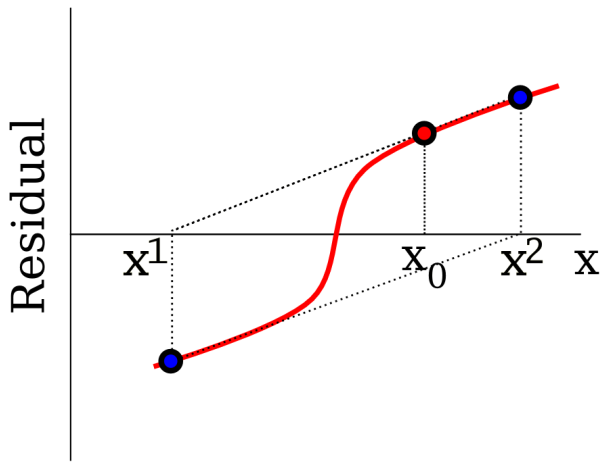
Iterations of Newton-Raphson method

$$R(x^1, f_1) + \left.\frac{\partial R}{\partial x}\right|_{x^1} \delta x = 0 \rightarrow \delta x = -\left.\frac{\partial R}{\partial x}\right|^{-1}_{x^1} R(x^1, f_1) \rightarrow x^2 = x^1 + \delta x$$

Particularities: mesh and convergence

- Strong **mesh refinement** is required
  - especially at **unknown edges** of contact zones
- **Slow change** of boundary conditions:
  - strong non-linearities of contact / friction problems
  - non-uniqueness of solution for frictional problems

Infinite looping

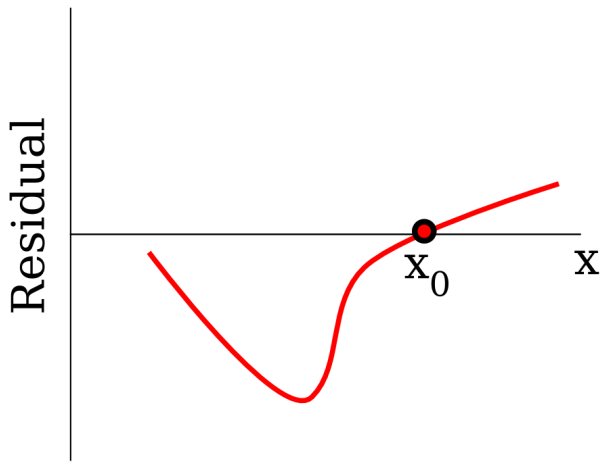


Infinite looping

Particularities: mesh and convergence

- Strong **mesh refinement** is required
  - especially at **unknown edges** of contact zones
- **Slow change** of boundary conditions:
  - strong non-linearities of contact / friction problems
  - non-uniqueness of solution for frictional problems

Convergence to a "false" solution

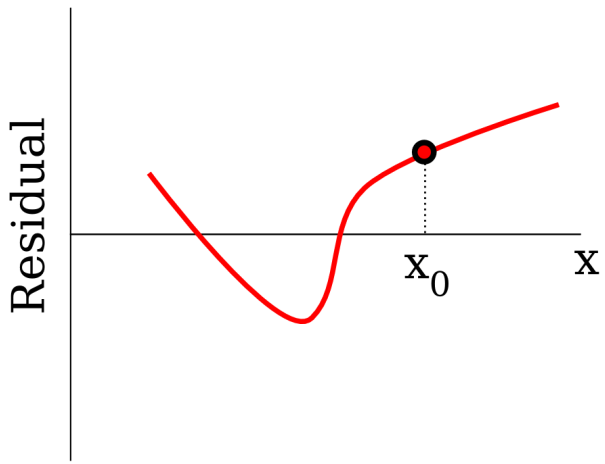


Initial guess $R(x_0, f_0) = 0$

Particularities: mesh and convergence

- Strong **mesh refinement** is required
  - especially at **unknown edges** of contact zones
- **Slow change** of boundary conditions:
  - strong non-linearities of contact / friction problems
  - non-uniqueness of solution for frictional problems

Convergence to a "false" solution

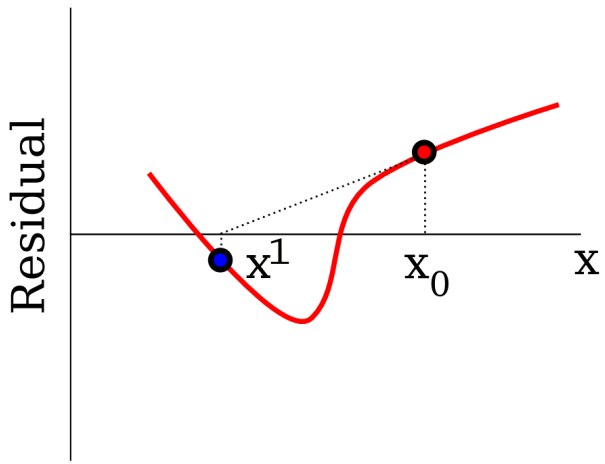


Too rapid change in boundary conditions $R(x_0, f_1) \neq 0$

Particularities: mesh and convergence

- Strong **mesh refinement** is required
  - especially at **unknown edges** of contact zones
- **Slow change** of boundary conditions:
  - strong non-linearities of contact / friction problems
  - non-uniqueness of solution for frictional problems

Convergence to a "false" solution



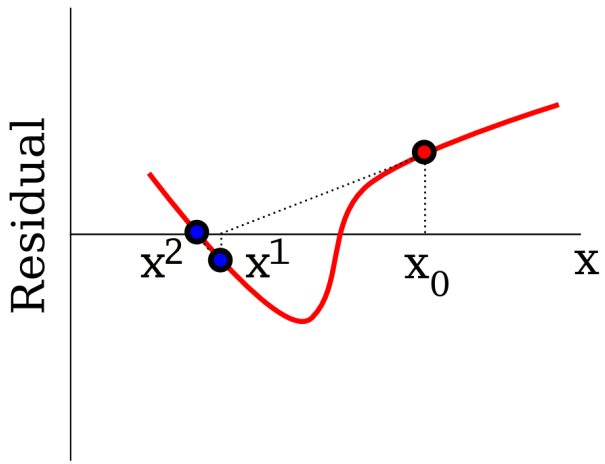
Iterations of Newton-Raphson method

$$R(x_0, f_1) + \left.\frac{\partial R}{\partial x}\right|_{x_0} \delta x = 0 \rightarrow \delta x = -\left.\frac{\partial R}{\partial x}\right|_{x_0}^{-1} R(x_0, f_1) \rightarrow x^1 = x_0 + \delta x$$

Particularities: mesh and convergence

- Strong **mesh refinement** is required
  - especially at **unknown edges** of contact zones
- **Slow change** of boundary conditions:
  - strong non-linearities of contact / friction problems
  - non-uniqueness of solution for frictional problems

Convergence to a "false" solution



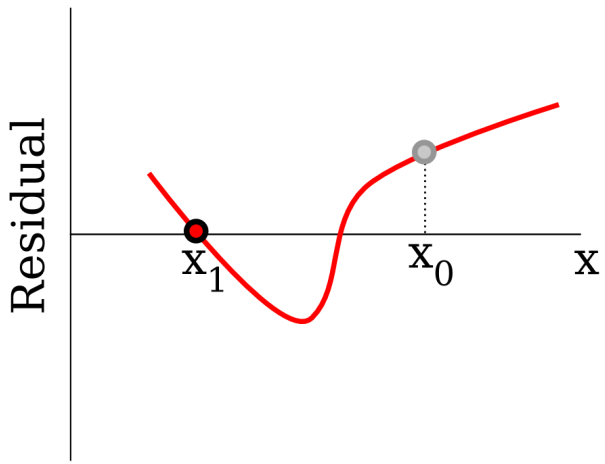
Iterations of Newton-Raphson method

$$R(x^1, f_1) + \left.\frac{\partial R}{\partial x}\right|_{x^1} \delta x = 0 \rightarrow \delta x = -\left.\frac{\partial R}{\partial x}\right|^{-1}_{x^1} R(x^1, f_1) \rightarrow x^2 = x^1 + \delta x$$

Particularities: mesh and convergence

- Strong **mesh refinement** is required
  - especially at **unknown edges** of contact zones
- **Slow change** of boundary conditions:
  - strong non-linearities of contact / friction problems
  - non-uniqueness of solution for frictional problems

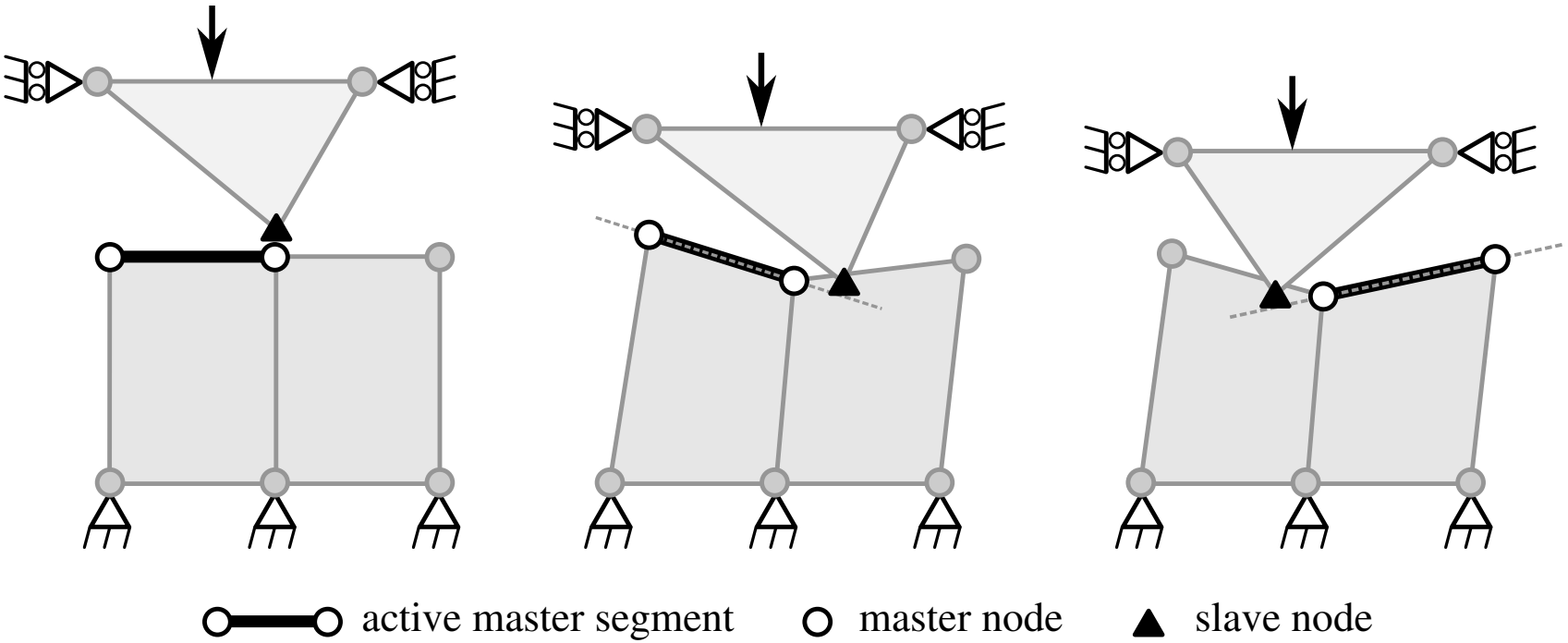
Convergence to a "false" solution



Convergence, but is it a "true" solution ?

Convergence problems: examples

- **Infinite looping, e.g.**

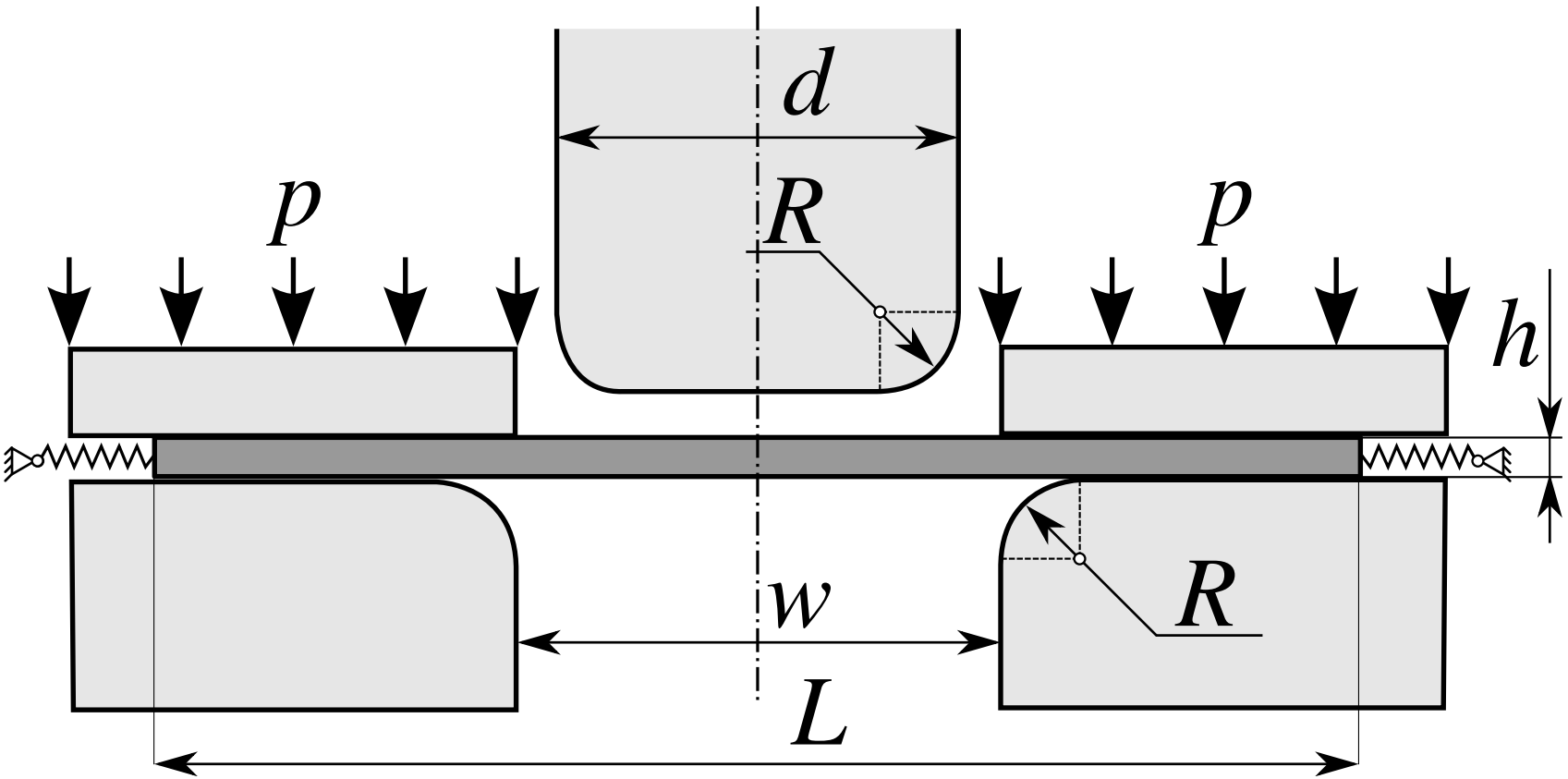


- Change of the contact state (contact/non-contact, stick/slip)
- Interplay between stiffness, friction and augmented Lagrangian coefficients[1]
- Combination of non-linearities (e.g., plasticity+contact)

Alart P., Méthode de Newton généralisée en mécanique du contact Journal de Mathématiques Pures et Appliqués 76 (1997)

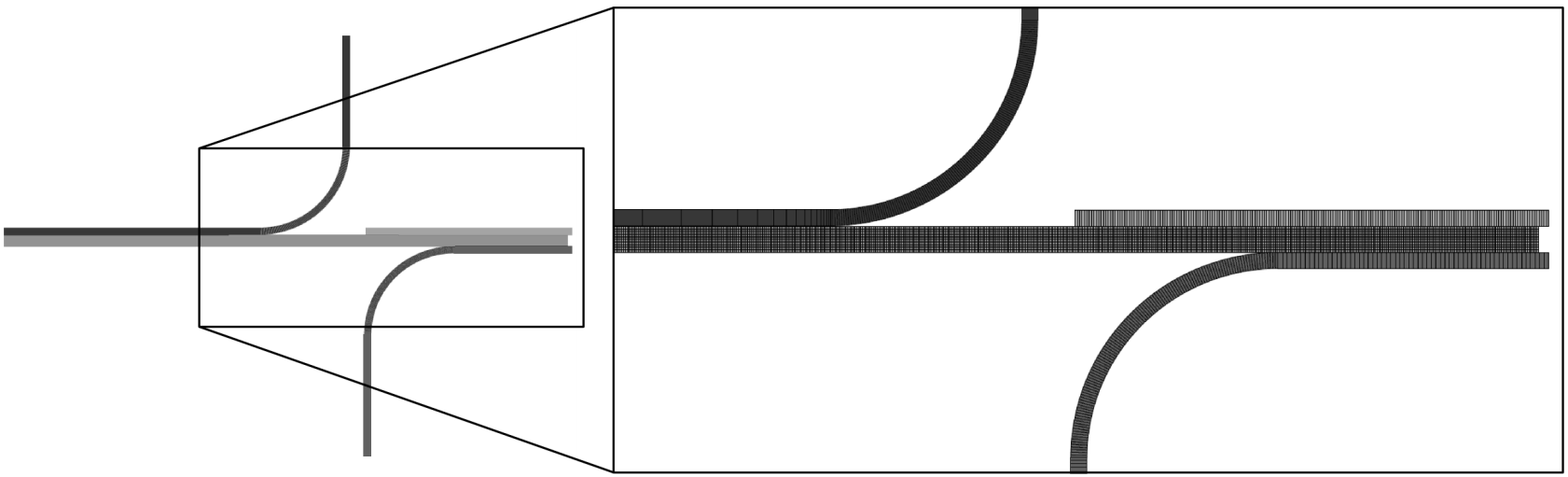
Convergence problems: examples

- Simulation of a deep drawing problem
- Dinite strain plasticity + frictional contact



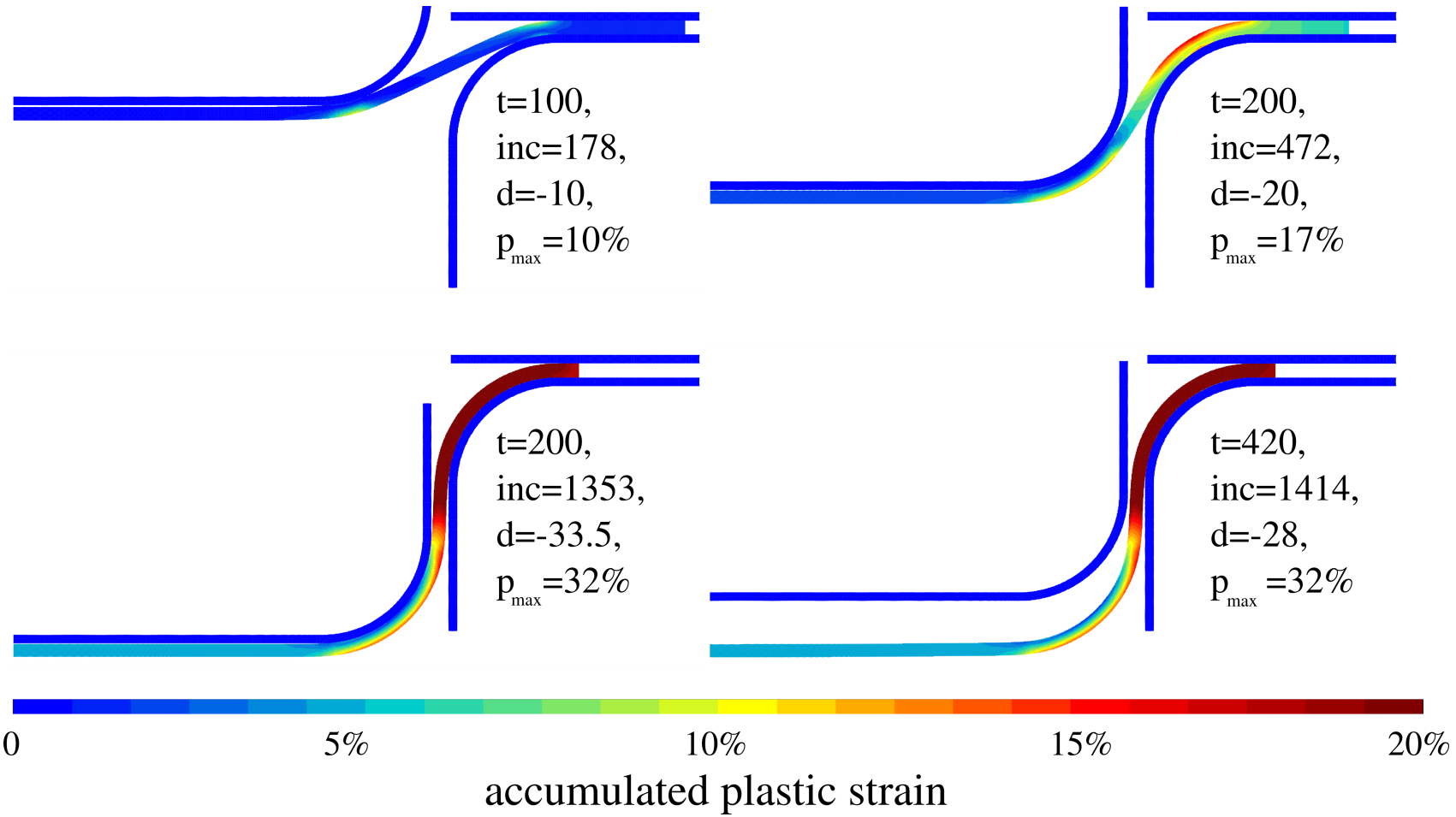
Convergence problems: examples

- Simulation of a deep drawing problem
- Dinite strain plasticity + frictional contact



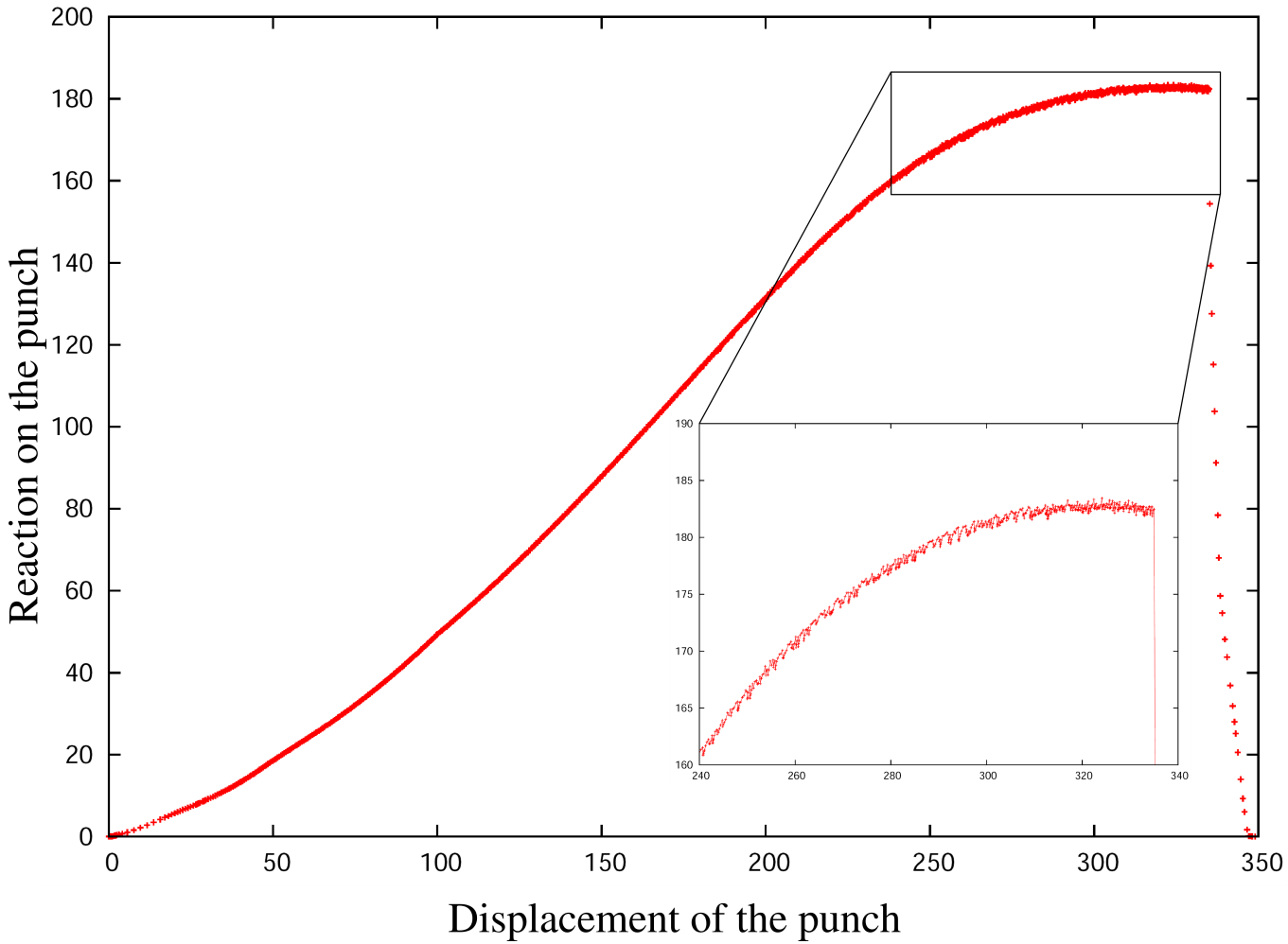
Convergence problems: examples

- Simulation of a deep drawing problem
- Dinite strain plasticity + frictional contact



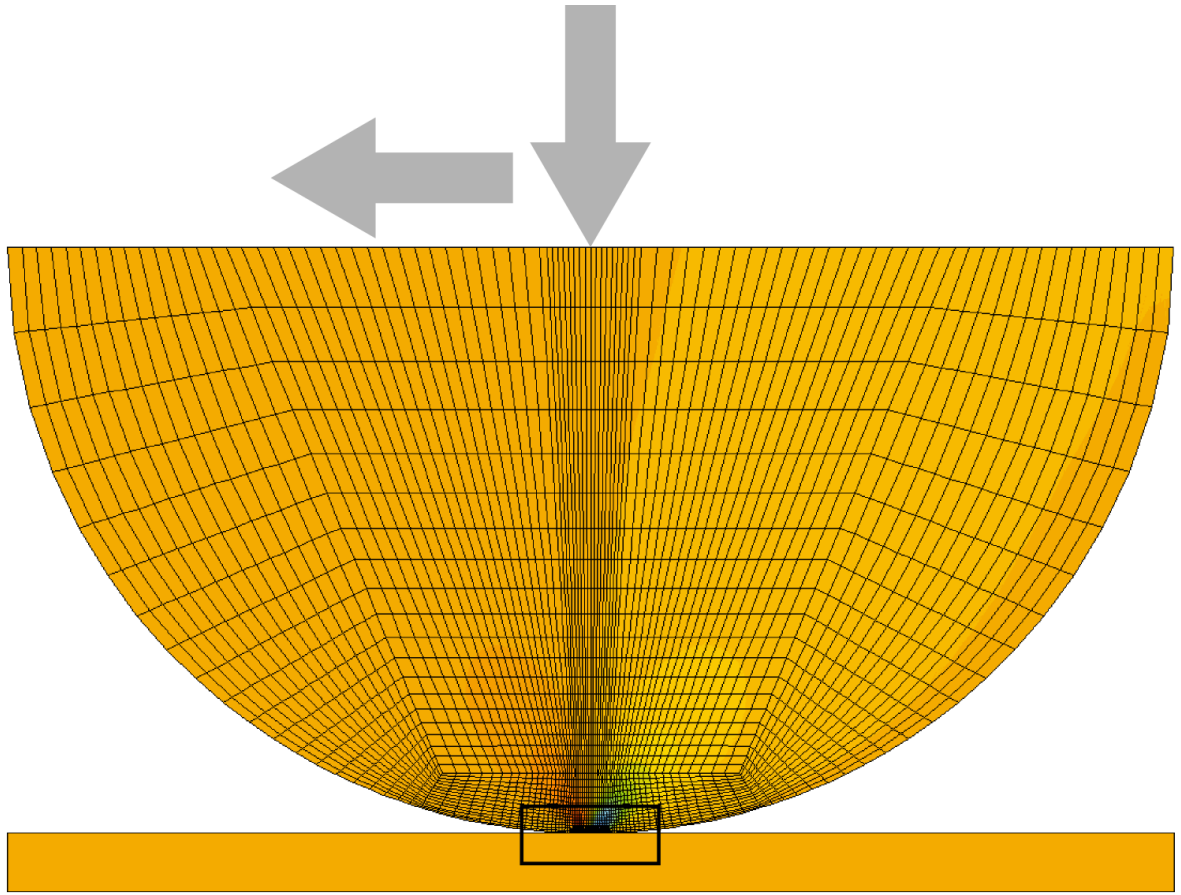
Convergence problems: examples

- Simulation of a deep drawing problem
- Dinite strain plasticity + frictional contact



Cylinder-plane frictional contact

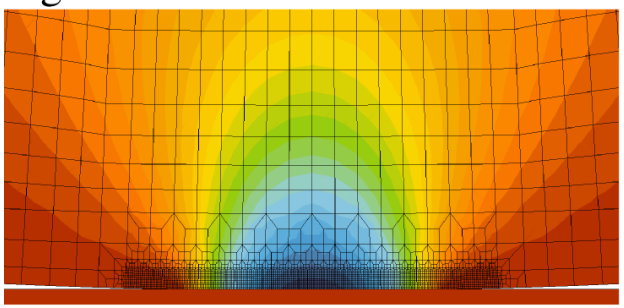
- Non-conservative problem, history of loading is crucial



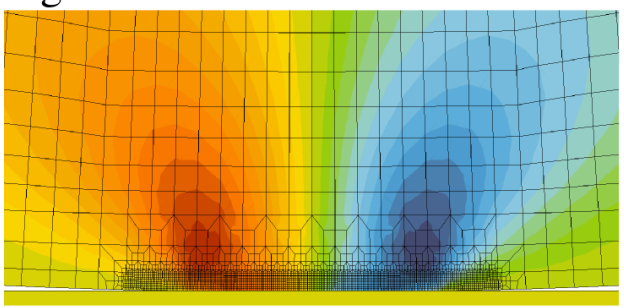
Cylinder-plane frictional contact

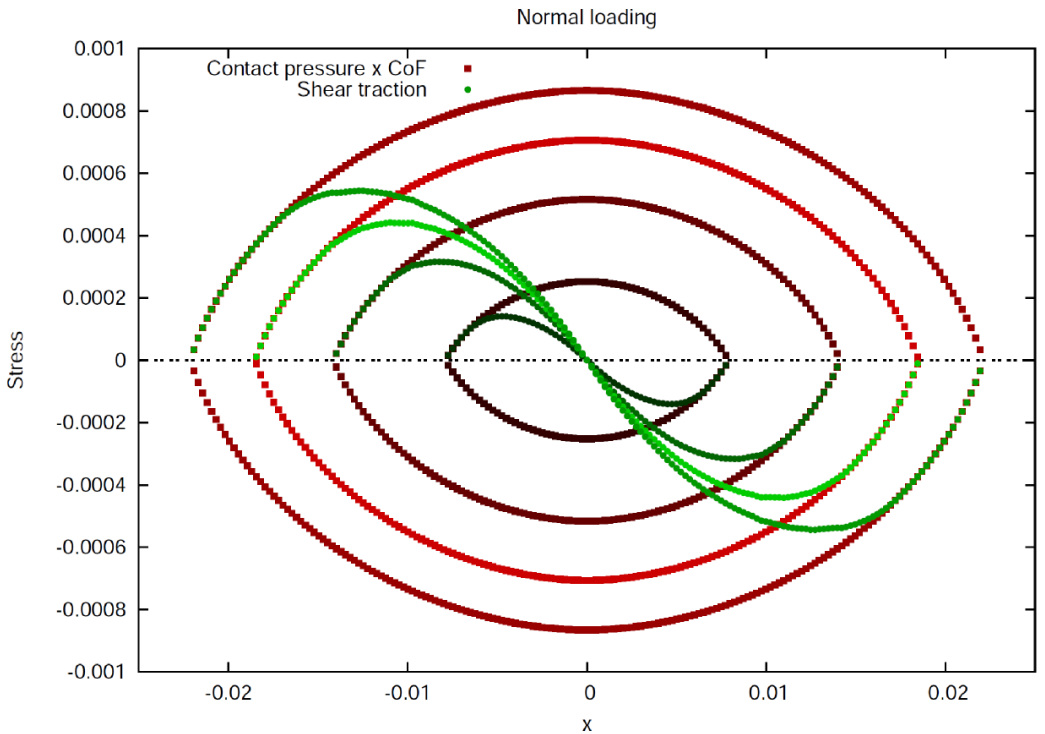
- Non-conservative problem, history of loading is crucial

sig22 at maximal normal load



sig12 at maximal normal load



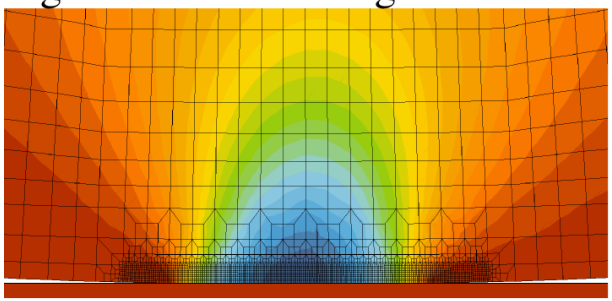


Press in 100 increments, $u_z \sim t^2$

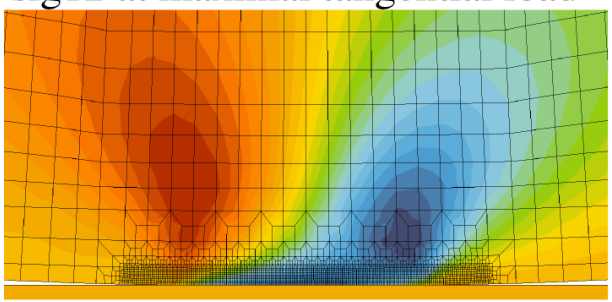
Cylinder-plane frictional contact

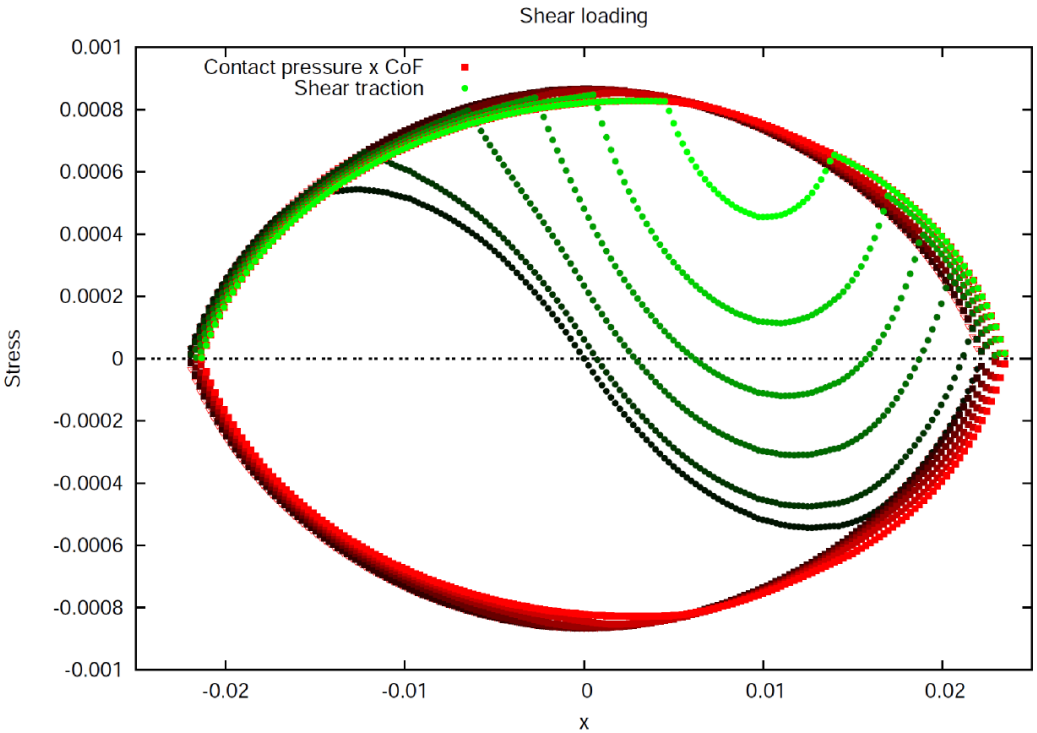
- Non-conservative problem, history of loading is crucial

sig22 at maximal tangential load



sig12 at maximal tangential load

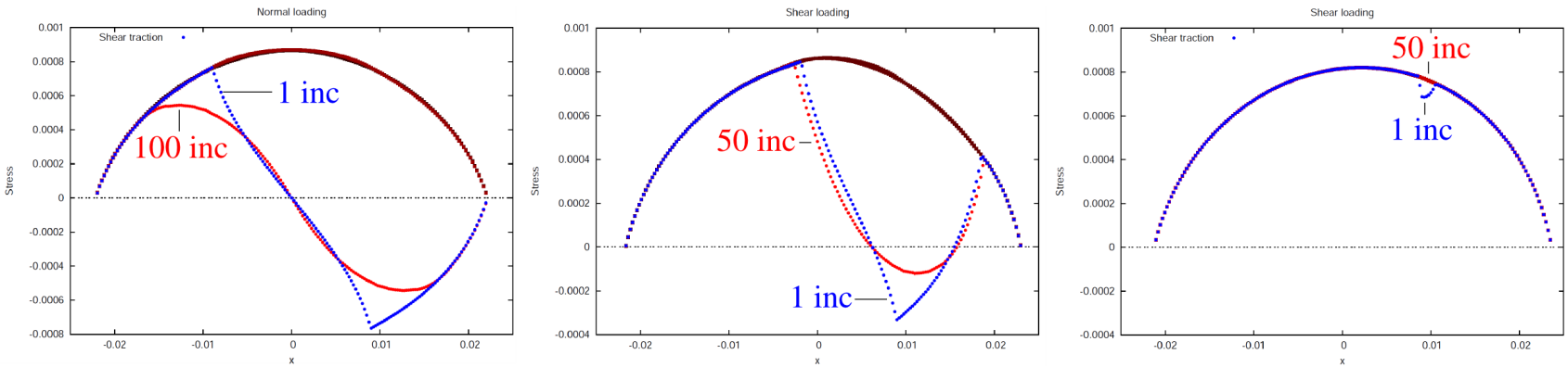




Shift in 100 increments, $u_z \sim t$

Cylinder-plane frictional contact

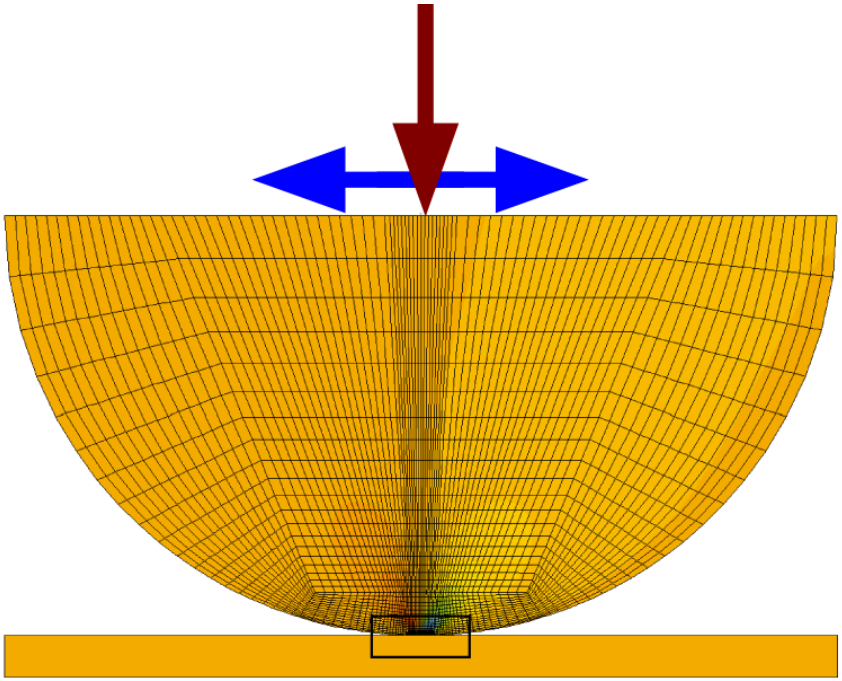
- Non-conservative problem, history of loading is crucial

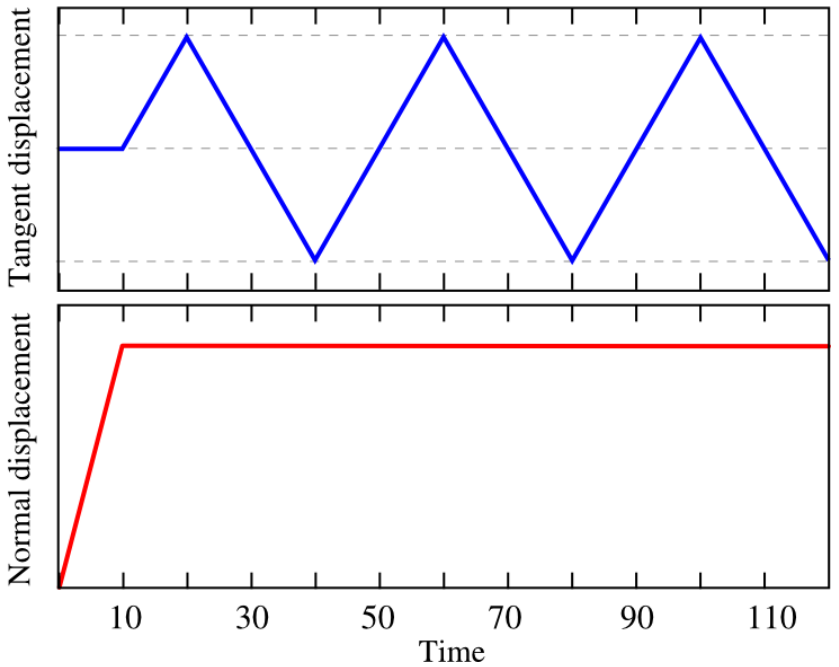


Comparison with: press in 1 increment, shift in 2 increments

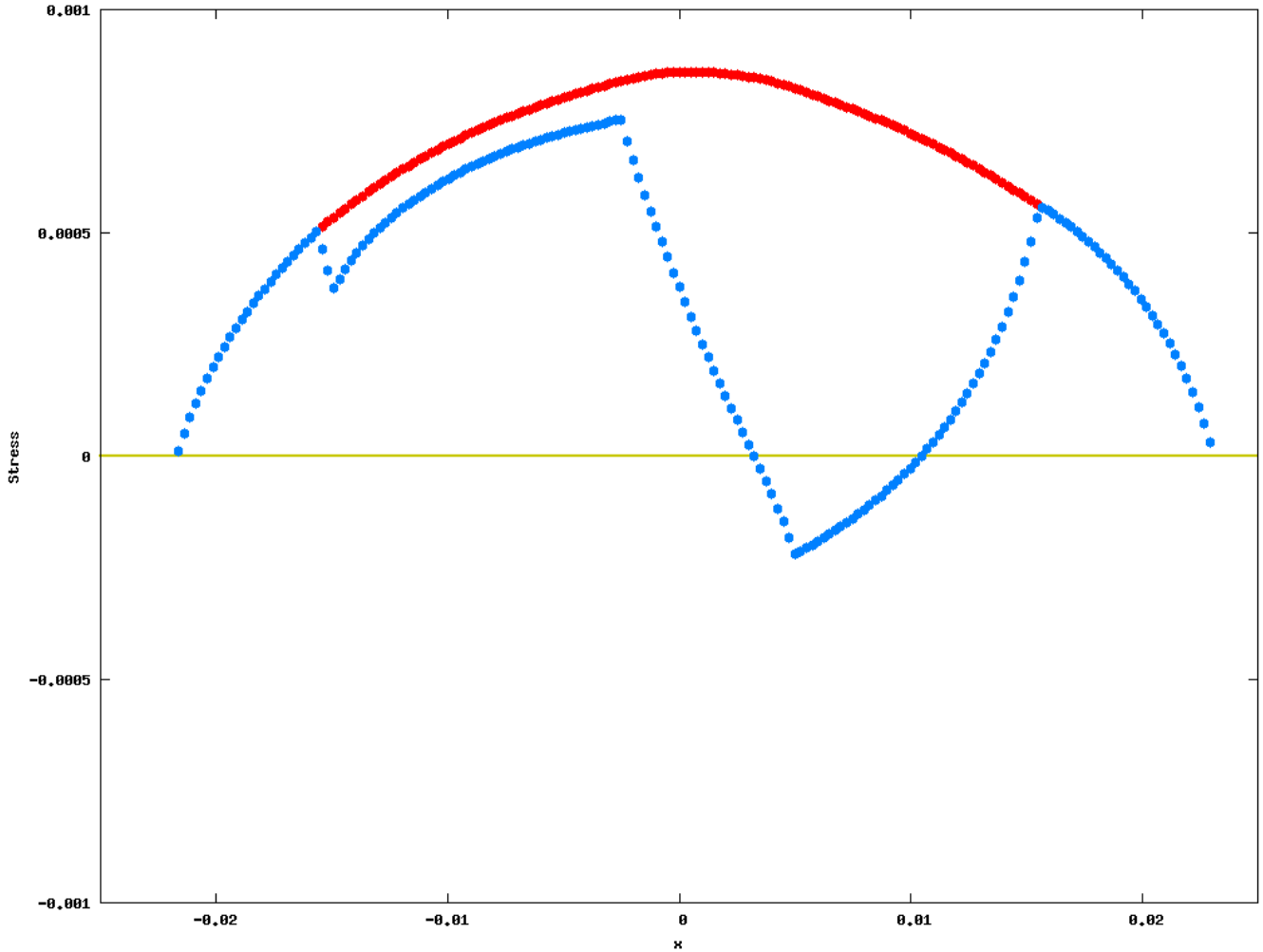
Before stick every point of the contact interface has to pass through the slip zone. It is impossible when loaded too fast.

Sphere-plane frictional contact: cycling

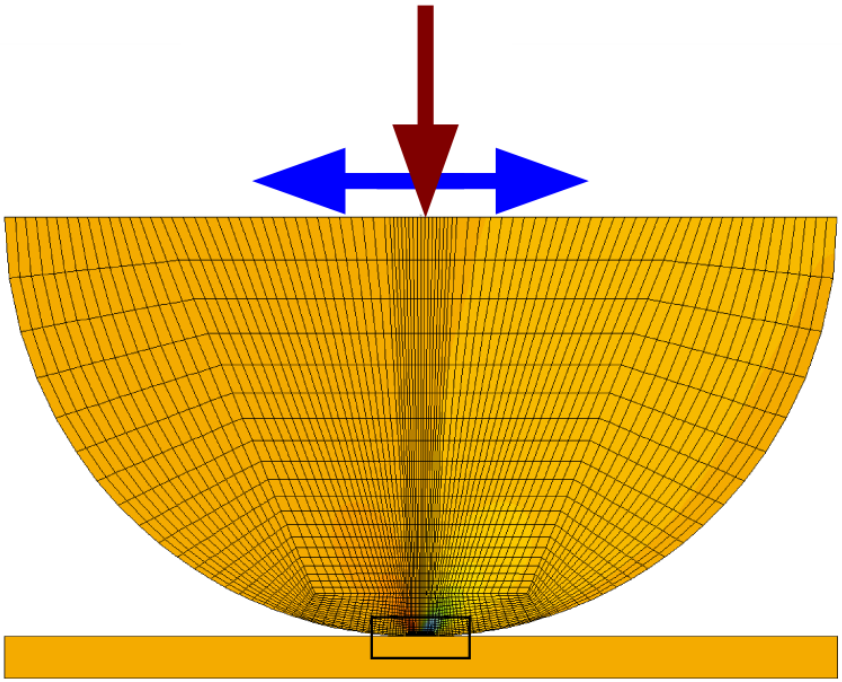


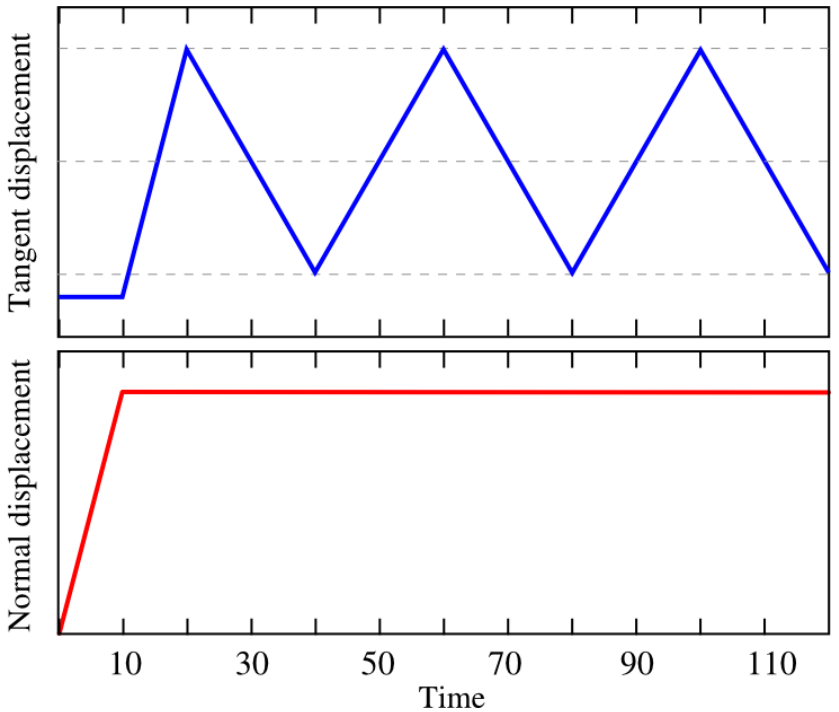


Sphere-plane frictional contact: cycling

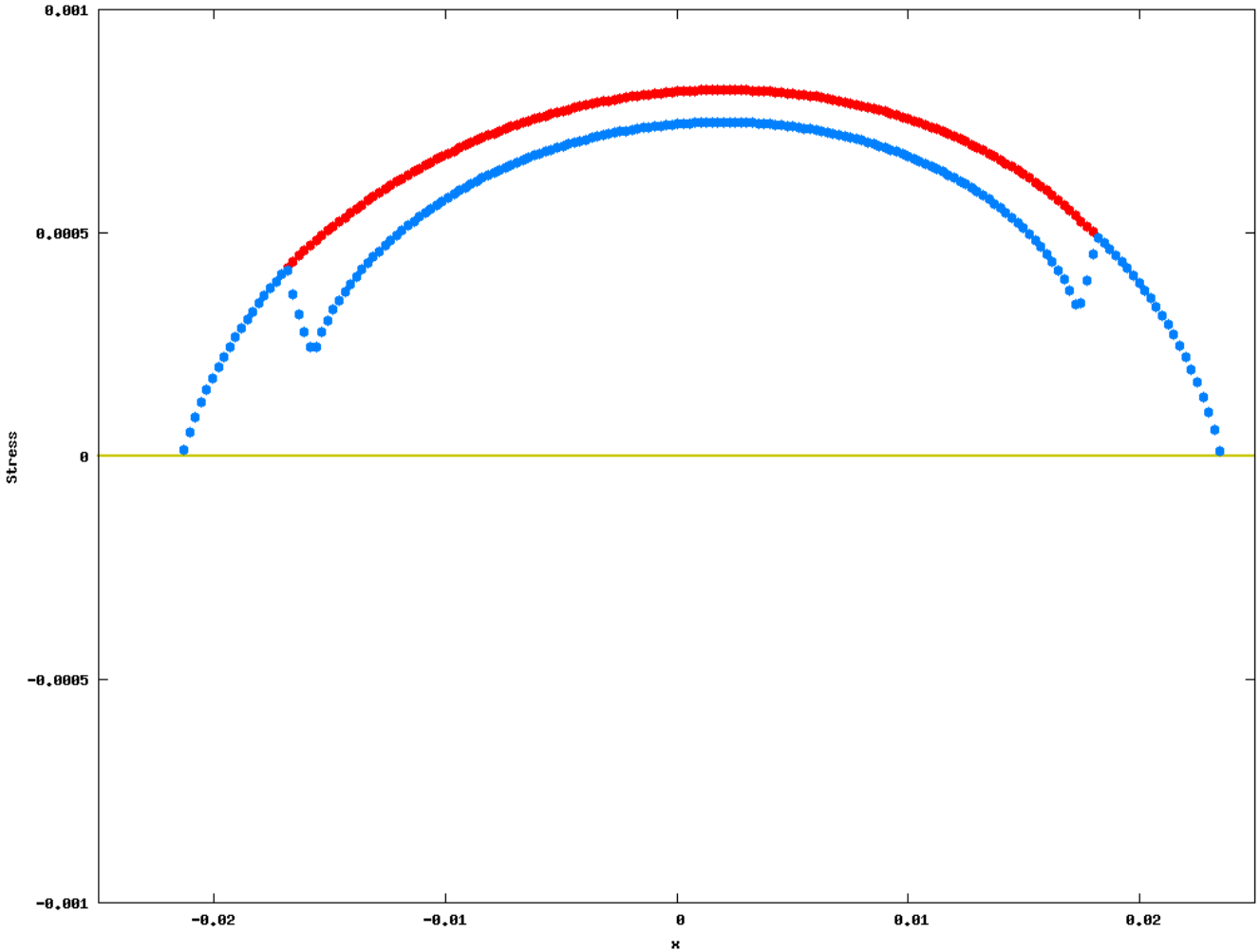


Sphere-plane frictional contact: cycling



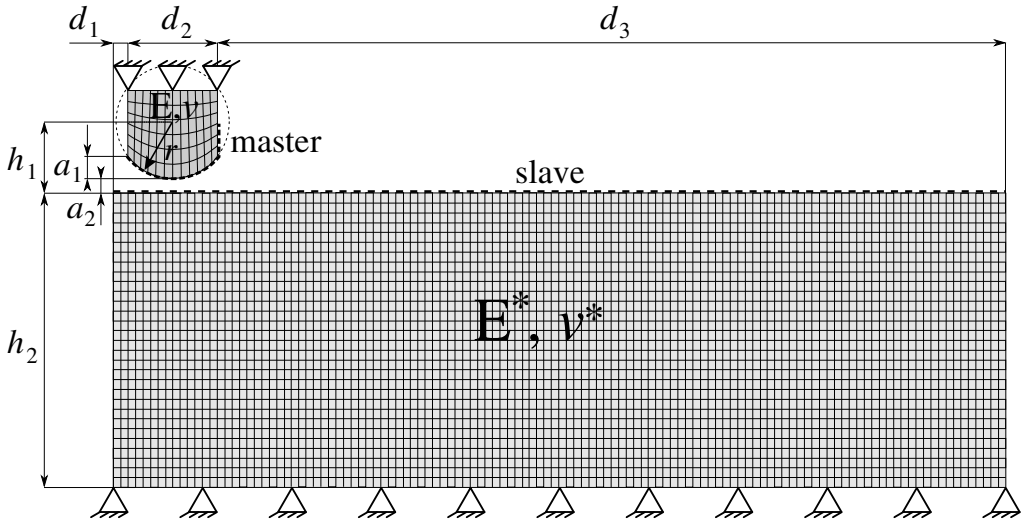


Sphere-plane frictional contact: cycling



Shallow ironing test

- Deformable-on-deformable frictional sliding
- Results obtained by different groups[1,2,3,4,5,6] differ significantly
- Local and global friction coefficients may differ



[1] Fischer K. A., Wriggers P., "Mortar based frictional contact formulation for higher order interpolations using the moving friction cone", Computer Methods in Applied Mechanics and Engineering, vol. 195, p. 5020-5036, 2006.

[2] Hartmann S., Oliver J., Cante J. C., Weyler R., Hernández J. A., "A contact domain method for large deformation frictional contact problems. Part 2: Numerical aspects", Computer Methods in Applied Mechanics and Engineering, vol. 198, p. 2607-2631, 2009.

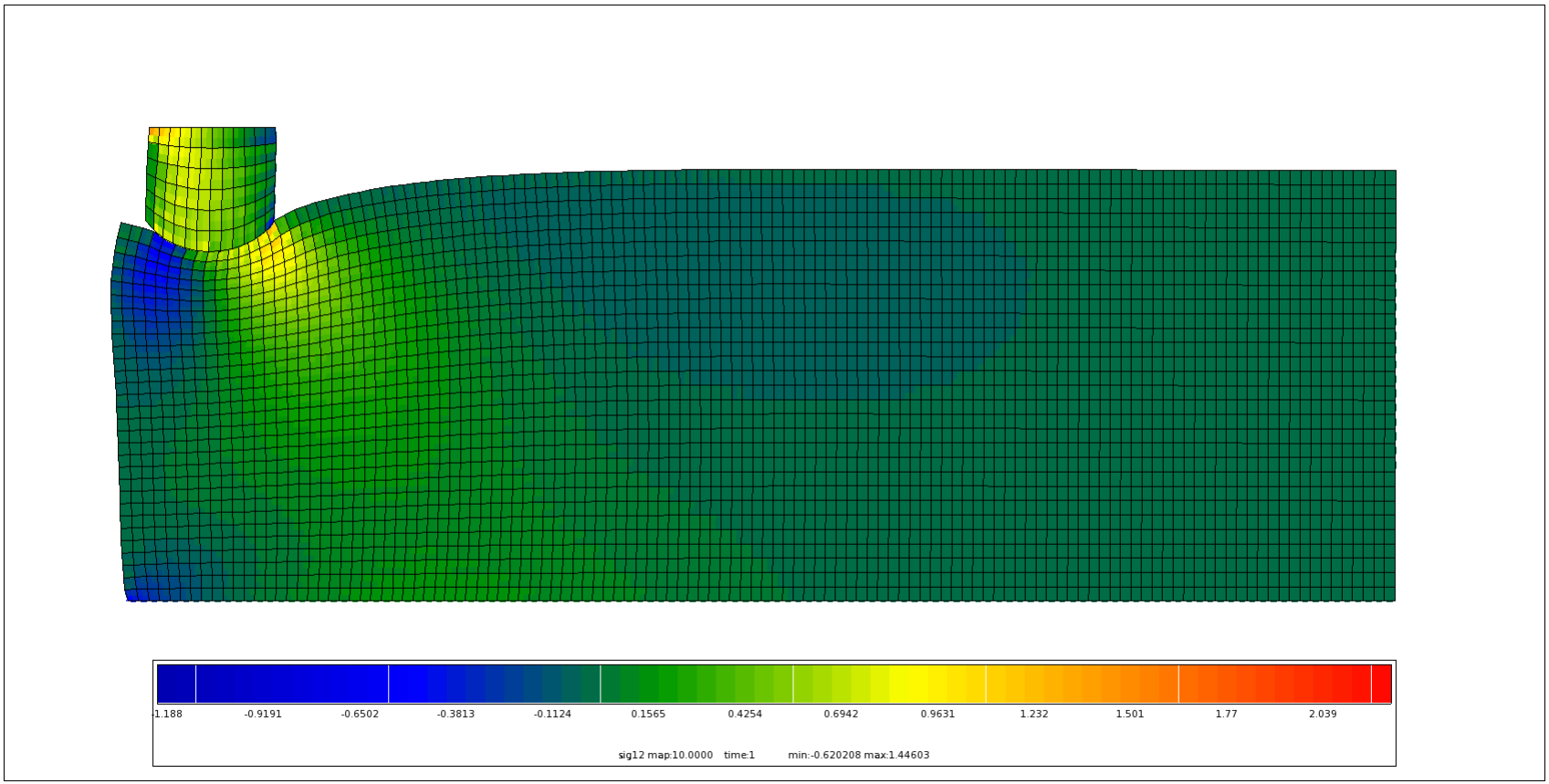
[3] Yastrebov V. A., "Computational contact mechanics: geometry, detection and numerical techniques", Thèse CdM & Onera, 2011.

[4] Kudawoo A. D., "Problèmes industriels de grande dimension en mécanique numérique du contact : performance, fiabilité et robustesse", Thèse @ LMA & LAMSID, 2012.

[5] Poulios K., Renard Y., "A non-symmetric integral approximation of large sliding frictional contact problems of deformable bodies based on ray-tracing", soumis, 2014.

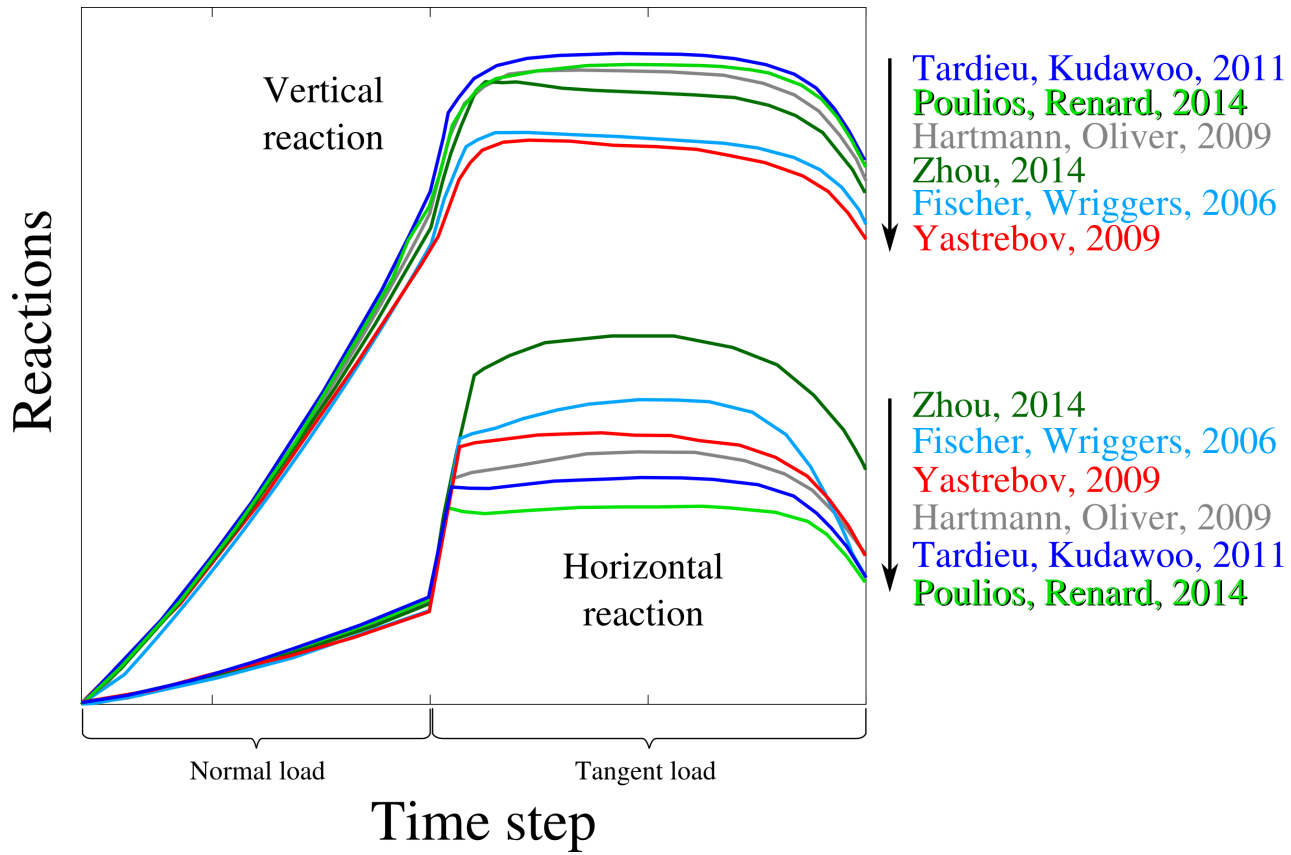
[6] Zhou Lei's blog, http://kt2008plus.blogspot.de

Shallow ironing test



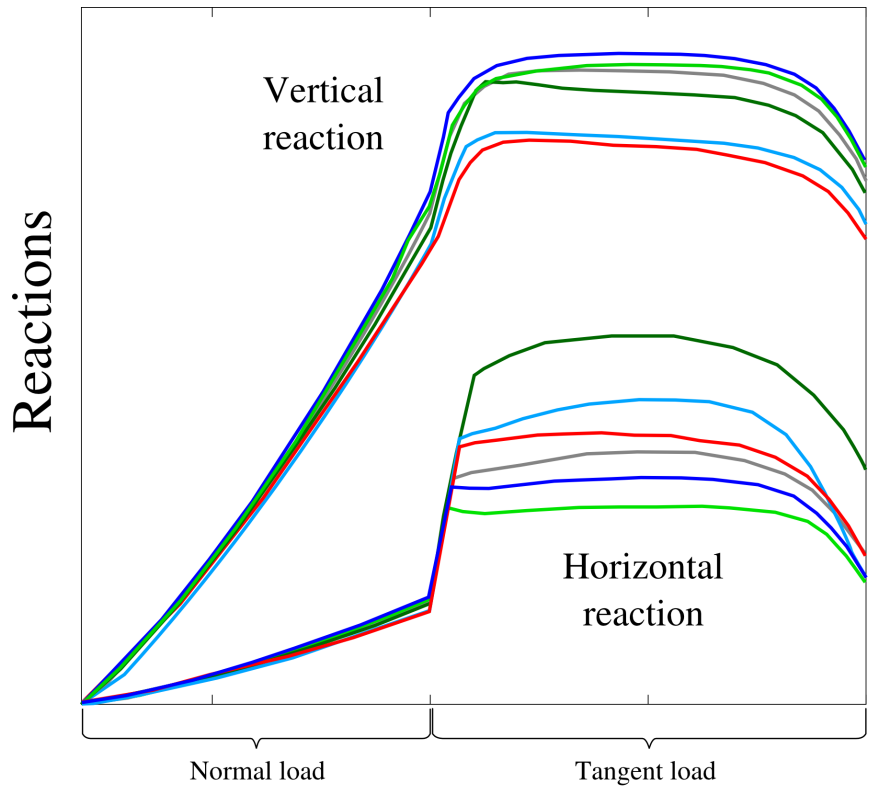
Shallow ironing test

- No agreement between authors
- Dif. authors used dif. meshes (quadrilateral lin./sq., triangular lin.)
- Dif. authors used either finite or infinitesimal strain formulation



Shallow ironing test

- No agreement between authors
- Dif. authors used dif. meshes (quadrilateral lin./sq., triangular lin.)
- Dif. authors used either finite or infinitesimal strain formulation



Global friction

0.59 Zhou, 2014

0.53 Fischer, Wriggers, 2006

0.47 Yastrebov, 2009

0.38 Hartmann, Oliver, 2009

0.34 Tardieu, Kudawoo, 2011

0.30 Poulios, Renard, 2014

Local coefficient of friction $\mu_l = 0.3$

Examples of contact problems

With analytical solution

- ⋆ linear elasticity
- ⋆ with/without friction

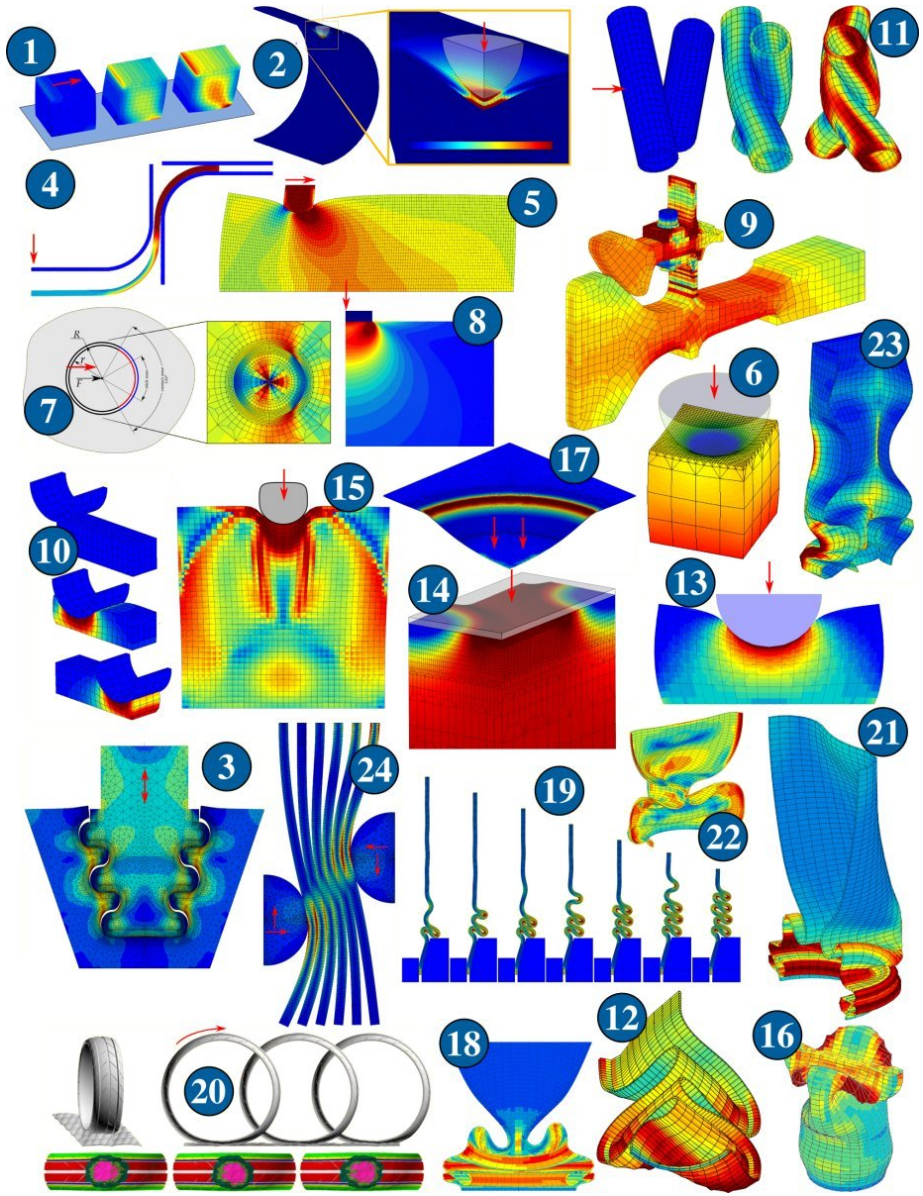
From literature

- ⋆ post-buckling 2D
- ⋆ finite strains
- ⋆ elasticity / plasticity
- ⋆ with/without friction

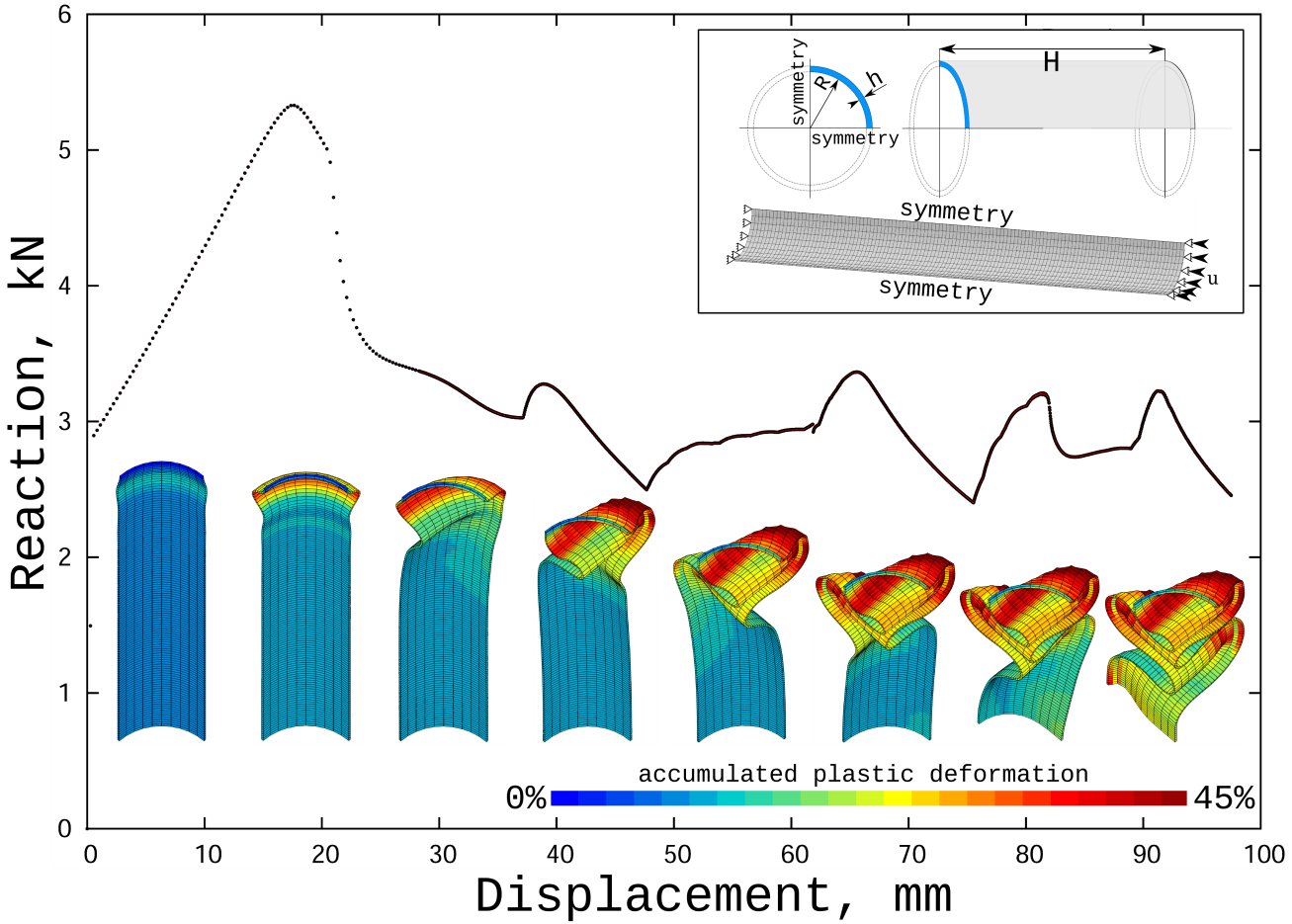
New

- ⋆ multi-contacts
- ⋆ post-buckling 3D
- ⋆ finite strains
- ⋆ elasticity / plasticity
- ⋆ with/without friction

Z-set Material & structure analysis suite



Self-contact problem



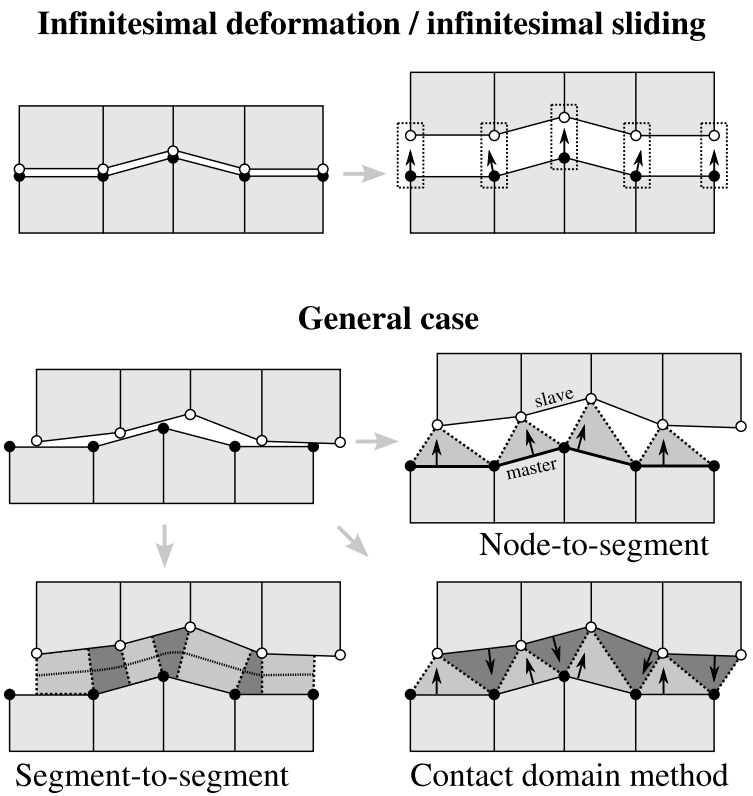
Finite element analysis of a post-buckling behavior of a thin walled tube

Collection of non-linearities: buckling instability, self-contact, finite strain plasticity

Z-set Material & structure analysis suite

Reading

- It's just a tip of the "Computational Contact Mechanics" iceberg
- Contact detection
- Contact discretization and integration
- Mortar & Nitshe method, contact in IGA
- Smoothing techniques
- Energy conservative methods for dynamics

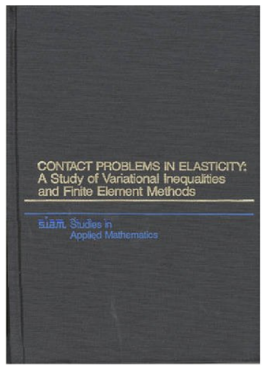


Contact discretization techniques

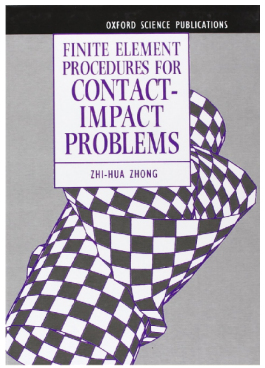
Reading

- It's just a tip of the "Computational Contact Mechanics" iceberg
- Contact detection
- Contact discretization and integration
- Mortar & Nitshe method, contact in IGA
- Smoothing techniques
- Energy conservative methods for dynamics

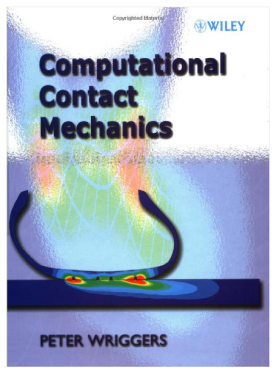
Kikuchi, Oden (1988)



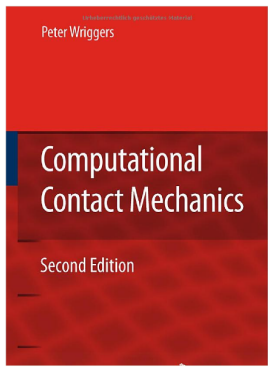
Zhong (1993)



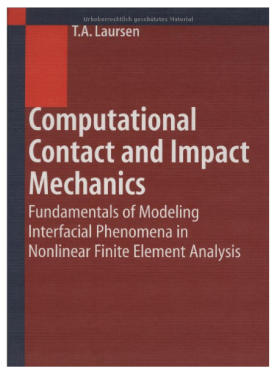
Wriggers (2002)



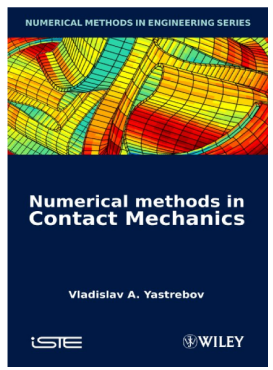
Wriggers, 2nd ed. (2006)



Laursen (2002)



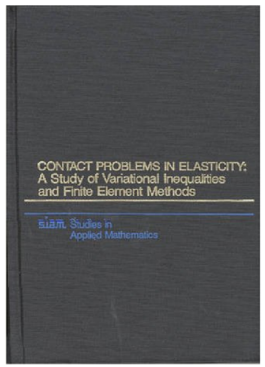
Yastrebov (2013)



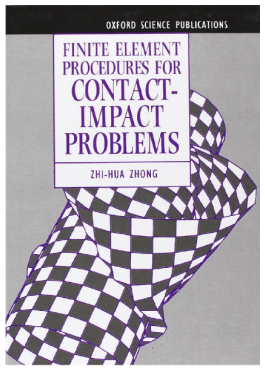
Reading

- It's just a tip of the "Computational Contact Mechanics" iceberg
- Contact detection
- Contact discretization and integration
- Mortar & Nitshe method, contact in IGA
- Smoothing techniques
- Energy conservative methods for dynamics
- Several advanced topics

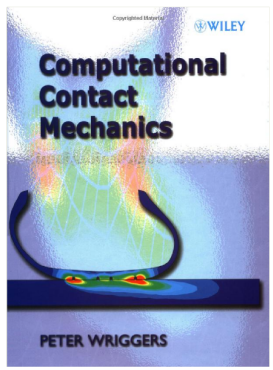
Kikuchi, Oden (1988)



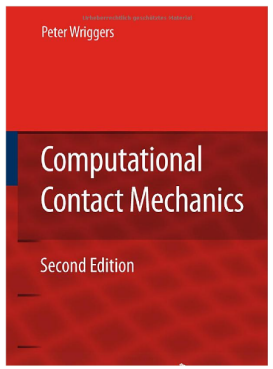
Zhong (1993)



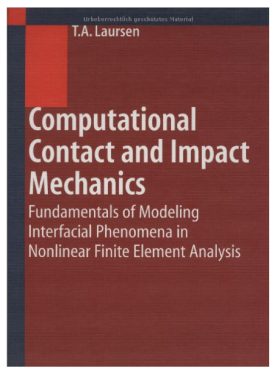
Wriggers (2002)



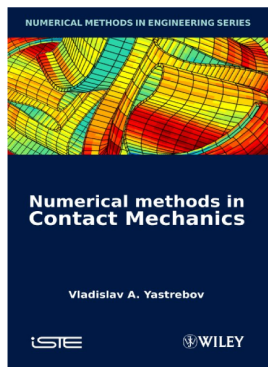
Wriggers, 2nd ed. (2006)



Laursen (2002)



Yastrebov (2013)



$\mathcal{L}_a(x, \lambda)$

Thank you for your attention!