# Flamant's problem 

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February 6, 2023

## 1 Problem statement

In Fig. 1 an elastic half-plane, Young's modulus $E$ and Poisson's ratio $v$, is shown. On its surface, a semi-circular groove of radius $r$ is loaded by a distributed pressure $p(\theta)=p_{0} \cos (\theta)$.

Problem: Find an induced stress state, deformation state and displacement field. Obtain asymptotic results for $r \rightarrow 0$ assuming that the resulting vertical force remains fixed.


Figure 1: Elastic half-plane with a semi-circular groove on its surface subject to a distributed pressure $p(\theta)$

## 2 Stress tensor distribution

The stress state is given by the following tensor in polar coordinates

$$
\begin{equation*}
\underline{\underline{\sigma}}=-\frac{\alpha \cos (\theta)}{r}\left(\underline{e}_{r} \otimes \underline{e}_{r}+v \underline{e}_{z} \otimes \underline{e}_{z}\right), \tag{1}
\end{equation*}
$$

where $\alpha=r_{0} p_{0}$. The integral of the stress vector over the circular hole gives:

$$
\begin{equation*}
-\int_{-\pi / 2}^{\pi / 2} \underline{\sigma} \cdot \underline{e}_{r} r_{0} d \theta=\frac{\alpha \pi}{2} \underline{e}_{y}=F \underline{e}_{y^{\prime}} \tag{2}
\end{equation*}
$$

then

$$
\begin{equation*}
\alpha=\frac{2 F}{\pi}, \tag{3}
\end{equation*}
$$

where $F$ is the linear density of applied normal force.

## 3 Strain tensor distribution

The strain tensor is given by

$$
\begin{equation*}
\underline{\underline{\varepsilon}}=-\frac{\alpha \cos (\theta)}{r E}\left[\left(1-v^{2}\right) \underline{e}_{r} \otimes \underline{e}_{r}-v(1+v) \underline{e}_{\theta} \otimes \underline{e}_{\theta}\right] \tag{4}
\end{equation*}
$$

## 4 Displacement field

The radial displacement can be found by integrating $\varepsilon_{r r}=\partial u_{r} / \partial r$ :

$$
\begin{equation*}
u_{r}=-\frac{\alpha \cos (\theta)\left(1-v^{2}\right)}{E} \log (r)+f(\theta), \tag{5}
\end{equation*}
$$

where $f(\theta)$ is an uknown function. The second displacement component $u_{\theta}$ can be found through the expression of $\varepsilon_{\theta \theta}=\frac{1}{r}\left(\partial u_{\theta} / \partial \theta+u_{r}\right)$, which after integration takes the form:

$$
\begin{equation*}
u_{\theta}=-\frac{\alpha \sin (\theta) v(1+v)}{E}+\frac{\alpha \sin (\theta)\left(1-v^{2}\right)}{E} \log (r)-\int f(\theta) d \theta+g(r) \tag{6}
\end{equation*}
$$

where $g(r)$ is another unknown function. So, we have two unknown functions and will need at least two equations to identify them. The both can be obtained from the fact that $\varepsilon_{r \theta}=0$, in polar coordinates it has a form:

$$
\begin{equation*}
\varepsilon_{r \theta}=\frac{1}{2}\left[\frac{1}{r}\left(\frac{\partial u_{r}}{\partial \theta}-u_{\theta}\right)+\frac{\partial u_{\theta}}{\partial r}\right]=0 \tag{7}
\end{equation*}
$$

or equvalently for non-zero $r$

$$
\begin{equation*}
\frac{\partial u_{r}}{\partial \theta}-u_{\theta}+r \frac{\partial u_{\theta}}{\partial r}=0 \tag{8}
\end{equation*}
$$

We substitute (5) and (6) in it and obtain:

$$
\begin{equation*}
\frac{\partial f(\theta)}{\partial \theta}+\frac{\alpha \sin (\theta) v(1+v)}{E}+\int f(\theta) d \theta-g(r)-\frac{\alpha \sin (\theta)\left(1-v^{2}\right)}{E}+r \frac{\partial g(r)}{\partial r}=0 \tag{9}
\end{equation*}
$$

After grouping terms that depend solely on $r$ and on $\theta$ we obtain the following equality:

$$
\begin{equation*}
\frac{\partial f(\theta)}{\partial \theta}+\int f(\theta) d \theta-\frac{\alpha \sin (\theta)(1+v)(1-2 v)}{E}=g(r)-r \frac{\partial g(r)}{\partial r} \tag{10}
\end{equation*}
$$

Thanks to this separation of variables, both the left and the right hand sides should be equal to the same constant $C$, and we obtain two equations needed to find $f(\theta)$ and $g(r)$ :

$$
\left\{\begin{array}{l}
\frac{\partial f(\theta)}{\partial \theta}+\int f(\theta) d \theta-\frac{\alpha \sin (\theta)(1+v)(1-2 v)}{E}=C  \tag{11}\\
g(r)-r \frac{\partial g(r)}{\partial r}=C
\end{array}\right.
$$

We take the derivative of the first and obtain:

$$
\begin{equation*}
\frac{\partial^{2} f(\theta)}{\partial \theta^{2}}+f(\theta)=\frac{\alpha \cos (\theta)(1+v)(1-2 v)}{E} \tag{12}
\end{equation*}
$$

The solution of the homogeneous (for zero right hand part) linear second-order differential equation is given by:

$$
\begin{equation*}
f_{0}(\theta)=A \cos (\theta)+B \sin (\theta) \tag{13}
\end{equation*}
$$

the particular solution we can seek in the form:

$$
\begin{equation*}
f_{*}(\theta)=h(\theta) \sin (\theta) \tag{14}
\end{equation*}
$$

which after its substitution in (12) gives:

$$
\begin{equation*}
\frac{\partial^{2} h}{\partial \theta^{2}} \sin (\theta)+2 \frac{\partial h}{\partial \theta} \cos (\theta)=\frac{\alpha \cos (\theta)(1+v)(1-2 v)}{E} \tag{15}
\end{equation*}
$$

therefore

$$
\begin{equation*}
\frac{\partial^{2} h}{\partial \theta^{2}}=0 \quad \text { and } \quad 2 \frac{\partial h}{\partial \theta}=\frac{\alpha(1+v)(1-2 v)}{E} \tag{16}
\end{equation*}
$$

since we have already $B \sin (\theta)$ in our solution of the homogeneous equation $f_{0}$, we keep only the linear term of function $h(\theta)=\alpha(1+v)(1-2 v) \theta /(2 E)$ :

$$
\begin{equation*}
f_{*}(\theta)=\frac{\alpha(1+v)(1-2 v)}{2 E} \theta \sin (\theta) \tag{17}
\end{equation*}
$$

The full solution for $f(\theta)$ is then given by:

$$
\begin{equation*}
f(\theta)=A \cos (\theta)+B \sin (\theta)+\frac{\alpha(1+v)(1-2 v)}{2 E} \theta \sin (\theta) . \tag{18}
\end{equation*}
$$

For the function $g(r)$, from Eq. (11) it immediately follows that

$$
\begin{equation*}
g(r)=E r+C \tag{19}
\end{equation*}
$$

Finally, the displacements are given by:

$$
u_{r}=-\frac{\alpha \cos (\theta)\left(1-v^{2}\right)}{E} \log (r)+\underbrace{A \cos (\theta)+B \sin (\theta)}_{\text {Rigid body displacement }}+\frac{\alpha(1+v)(1-2 v)}{2 E} \theta \sin (\theta)
$$

$$
\begin{gather*}
u_{\theta}=-\frac{\alpha \sin (\theta) v(1+v)}{E}+\frac{\alpha \sin (\theta)\left(1-v^{2}\right)}{E} \log (r) \underbrace{-A \sin (\theta)+B \cos (\theta)}_{\text {Rigid body displacement }}-\frac{\alpha(1+v)(1-2 v)}{2 E} \sin (\theta)+ \\
+\frac{\alpha(1+v)(1-2 v)}{2 E} \theta \cos (\theta)+\underbrace{E r}_{\text {Rigid body rotation }}+C
\end{gather*}
$$

If we remove rigid body motion, we obtain the following displacements on the surface:

$$
\begin{gather*}
u_{x}=-\frac{F(1+v)(1-2 v)}{2 E} \operatorname{sign}(x)  \tag{22}\\
u_{y}=\frac{2 F\left(1-v^{2}\right)}{\pi E} \log (|x|)+C \tag{23}
\end{gather*}
$$

Note that $u_{x}=u_{r} \underline{e}_{r} \cdot \underline{e}_{x}$ for $\theta= \pm \pi / 2$, and $u_{y}=u_{\theta} \underline{e}_{\theta} \cdot \underline{e}_{y}$ for $\theta= \pm \pi / 2$. We also used the expression for $\alpha$ from Eq. (3).

