Flamant's problem

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1 Problem statement

In Fig. 1 an elastic half-plane, Young's modulus *E* and Poisson's ratio *v*, is shown. On its surface, a semi-circular groove of radius *r* is loaded by a distributed pressure $p(\theta) = p_0 \cos(\theta)$.

Problem: Find an induced stress state, deformation state and displacement field. Obtain asymptotic results for $r \rightarrow 0$ assuming that the resulting vertical force remains fixed.



Figure 1: Elastic half-plane with a semi-circular groove on its surface subject to a distributed pressure $p(\theta)$

2 Stress tensor distribution

The stress state is given by the following tensor in polar coordinates

$$\underline{\underline{\sigma}} = -\frac{\alpha \cos(\theta)}{r} \left(\underline{\underline{e}}_r \otimes \underline{\underline{e}}_r + \nu \underline{\underline{e}}_z \otimes \underline{\underline{e}}_z \right),\tag{1}$$

where $\alpha = r_0 p_0$. The integral of the stress vector over the circular hole gives:

$$-\int_{-\pi/2}^{\pi/2} \underline{\underline{\sigma}} \cdot \underline{\underline{e}}_r r_0 d\theta = \frac{\alpha \pi}{2} \underline{\underline{e}}_y = F \underline{\underline{e}}_{y'}$$
(2)

then

$$\alpha = \frac{2F}{\pi},\tag{3}$$

where *F* is the linear density of applied normal force.

3 Strain tensor distribution

The strain tensor is given by

$$\underline{\underline{\varepsilon}} = -\frac{\alpha \cos(\theta)}{rE} \left[(1 - \nu^2) \underline{\underline{e}}_r \otimes \underline{\underline{e}}_r - \nu (1 + \nu) \underline{\underline{e}}_\theta \otimes \underline{\underline{e}}_\theta \right]$$
(4)

4 Displacement field

The radial displacement can be found by integrating $\varepsilon_{rr} = \partial u_r / \partial r$:

$$u_r = -\frac{\alpha \cos(\theta)(1-\nu^2)}{E}\log(r) + f(\theta),$$
(5)

where $f(\theta)$ is an uknown function. The second displacement component u_{θ} can be found through the expression of $\varepsilon_{\theta\theta} = \frac{1}{r}(\partial u_{\theta}/\partial \theta + u_r)$, which after integration takes the form:

$$u_{\theta} = -\frac{\alpha \sin(\theta)\nu(1+\nu)}{E} + \frac{\alpha \sin(\theta)(1-\nu^2)}{E}\log(r) - \int f(\theta)d\theta + g(r), \quad (6)$$

where g(r) is another unknown function. So, we have two unknown functions and will need at least two equations to identify them. The both can be obtained from the fact that $\varepsilon_{r\theta} = 0$, in polar coordinates it has a form:

$$\varepsilon_{r\theta} = \frac{1}{2} \left[\frac{1}{r} \left(\frac{\partial u_r}{\partial \theta} - u_\theta \right) + \frac{\partial u_\theta}{\partial r} \right] = 0, \tag{7}$$

or equvalently for non-zero r

$$\frac{\partial u_r}{\partial \theta} - u_\theta + r \frac{\partial u_\theta}{\partial r} = 0.$$
(8)

We substitute (5) and (6) in it and obtain:

$$\frac{\partial f(\theta)}{\partial \theta} + \frac{\alpha \sin(\theta)\nu(1+\nu)}{E} + \int f(\theta)d\theta - g(r) - \frac{\alpha \sin(\theta)(1-\nu^2)}{E} + r\frac{\partial g(r)}{\partial r} = 0.$$
(9)

After grouping terms that depend solely on *r* and on θ we obtain the following equality:

$$\frac{\partial f(\theta)}{\partial \theta} + \int f(\theta) d\theta - \frac{\alpha \sin(\theta)(1+\nu)(1-2\nu)}{E} = g(r) - r \frac{\partial g(r)}{\partial r}.$$
 (10)

Thanks to this separation of variables, both the left and the right hand sides should be equal to the same constant *C*, and we obtain two equations needed to find $f(\theta)$ and g(r):

$$\begin{cases} \frac{\partial f(\theta)}{\partial \theta} + \int f(\theta) d\theta - \frac{\alpha \sin(\theta)(1+\nu)(1-2\nu)}{E} = C\\ g(r) - r \frac{\partial g(r)}{\partial r} = C \end{cases}$$
(11)

We take the derivative of the first and obtain:

$$\frac{\partial^2 f(\theta)}{\partial \theta^2} + f(\theta) = \frac{\alpha \cos(\theta)(1+\nu)(1-2\nu)}{E}.$$
 (12)

The solution of the homogeneous (for zero right hand part) linear second-order differential equation is given by:

$$f_0(\theta) = A\cos(\theta) + B\sin(\theta), \tag{13}$$

the particular solution we can seek in the form:

$$f_*(\theta) = h(\theta)\sin(\theta), \tag{14}$$

which after its substitution in (12) gives:

$$\frac{\partial^2 h}{\partial \theta^2} \sin(\theta) + 2\frac{\partial h}{\partial \theta} \cos(\theta) = \frac{\alpha \cos(\theta)(1+\nu)(1-2\nu)}{E},$$
(15)

therefore

$$\frac{\partial^2 h}{\partial \theta^2} = 0 \quad \text{and} \quad 2\frac{\partial h}{\partial \theta} = \frac{\alpha(1+\nu)(1-2\nu)}{E},$$
 (16)

since we have already $B\sin(\theta)$ in our solution of the homogeneous equation f_0 , we keep only the linear term of function $h(\theta) = \alpha(1+\nu)(1-2\nu)\theta/(2E)$:

$$f_*(\theta) = \frac{\alpha(1+\nu)(1-2\nu)}{2E}\theta\sin(\theta).$$
(17)

The full solution for $f(\theta)$ is then given by:

$$f(\theta) = A\cos(\theta) + B\sin(\theta) + \frac{\alpha(1+\nu)(1-2\nu)}{2E}\theta\sin(\theta)$$
 (18)

For the function g(r), from Eq. (11) it immediately follows that

$$g(r) = Er + C. \tag{19}$$

Finally, the displacements are given by:

$$u_r = -\frac{\alpha \cos(\theta)(1-\nu^2)}{E}\log(r) + \underbrace{A\cos(\theta) + B\sin(\theta)}_{\text{Rigid body displacement}} + \frac{\alpha(1+\nu)(1-2\nu)}{2E}\theta\sin(\theta)$$
(20)

$$u_{\theta} = -\frac{\alpha \sin(\theta)v(1+v)}{E} + \frac{\alpha \sin(\theta)(1-v^2)}{E}\log(r) \qquad \underbrace{-A\sin(\theta) + B\cos(\theta)}_{\text{Rigid body displacement}} - \frac{\alpha(1+v)(1-2v)}{2E}\sin(\theta) + \underbrace{-A\sin(\theta) + B\cos(\theta)}_{\text{Rigid body displacement}} + \frac{\alpha(1+v)(1-2v)}{2E}\theta\cos(\theta) + \underbrace{-Er}_{\text{Rigid body rotation}} + C$$

(21)

If we remove rigid body motion, we obtain the following displacements on the surface:

$$u_x = -\frac{F(1+\nu)(1-2\nu)}{2E} \operatorname{sign}(x)$$
(22)

$$u_y = \frac{2F(1-\nu^2)}{\pi E} \log(|x|) + C$$
(23)

Note that $u_x = u_r \underline{e}_r \cdot \underline{e}_x$ for $\theta = \pm \pi/2$, and $u_y = u_\theta \underline{e}_\theta \cdot \underline{e}_y$ for $\theta = \pm \pi/2$. We also used the expression for α from Eq. (3).