Computational Contact Mechanics
Finite Element Method

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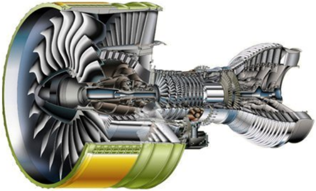
Athens week @ MINES Paris
March 16, 2023

Outline

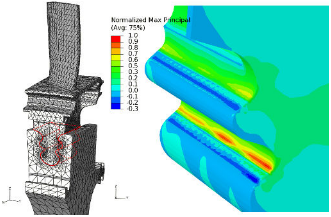
- Introduction
- Basics of Contact and Friction
- Towards a weak form
- Optimization methods
- Resolution algorithm
- Examples

Introduction

1. Assembled parts, e.g. engines



Aircraft's engine GP 7200
www.safran-group.com

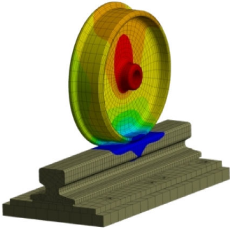


[1] M. W. R. Savage
J. Eng. Gas Turb. Power, 134:012501 (2012)

Industrial and natural contact problems

1. Assembled parts, e.g. engines
2. Railroad contacts

High speed train TGV www.sncf.com

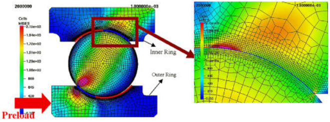


Wilde/ANSYS wildeanalysis.co.uk

Industrial and natural contact problems

1. Assembled parts, e.g. engines
2. Railroad contacts
3. Gears and bearings

Bearings
www.skf.com



[1] F. Massi, J. Rocchi, A. Culla, Y. Berthier Mech. Syst. Signal Pr., 24:1068-1080 (2010)

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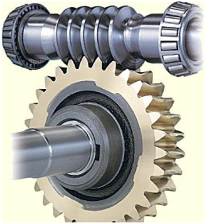
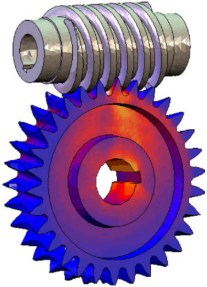


Helical gear www.tpg.com.tw



www.mscsoftware.com

Industrial and natural contact problems

1. Assembled parts, e.g. engines
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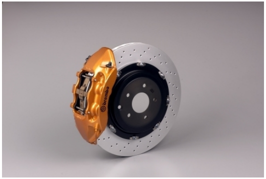
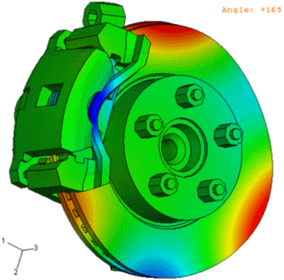
3. Gears and bearings
4. Breaking systems



Assembled breaking system

www.brembo.com



www.mechanicalengineeringblog.com

1. Assembled parts, e.g. engines
2. Railroad contacts
3. Gears and bearings
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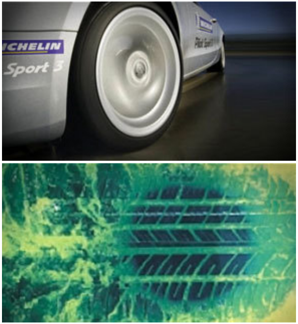
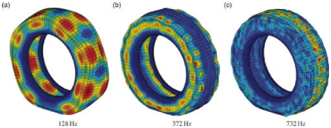
5. Tire-road contact



Tire-road contact www.michelin.com



[1] M. Brinkmeier, U. Nackenhorst, S. Petersen, O. von Estorff, J. Sound Vib., 309:20-39 (2008)

Industrial and natural contact problems

1. Assembled parts, e.g. engines
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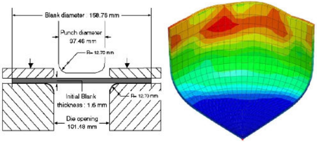
5. Tire-road contact
6. Metal forming

Deep drawing www.thomasnet.com



[1] G. Rousselier, F. Barlat, J. W. Yoon
Int. J. Plasticity, 25:2383-2409 (2009)

Industrial and natural contact problems

1. Assembled parts, e.g. engines
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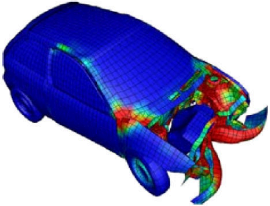
6. Metal forming
7. Crash tests

Crash-test www.porsche.com



[1] O. Klyavin, A. Michailov, A. Borovkov
www.fea.ru

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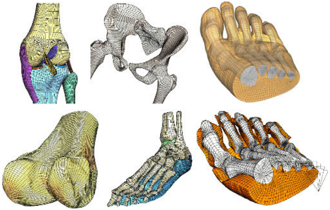
7. Crash tests
8. Biomechanics

Human articulations
www.sportssupplements.net



J. A. Weiss, University of Utah
Musculoskeletal Research Laboratories

Industrial and natural contact problems

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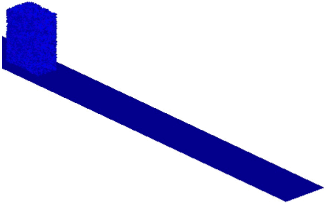
8. Biomechanics
9. Granular materials

Sand dunes www.en.wikipedia.org



E. Azema et al, LMGC90

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7. Crash tests
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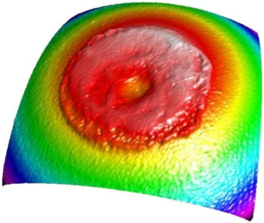
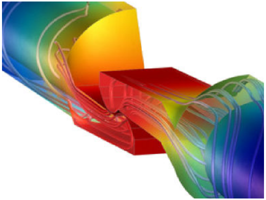
10. Electric contacts



Damage at electric contact zone
www.taicaan.com



Simulation of electric current
www.comsol.com

Industrial and natural contact problems

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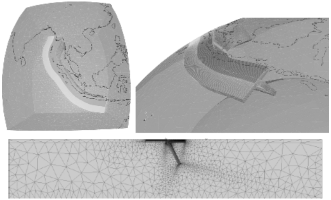
10. Electric contacts
11. Tectonic motions

San-Andreas fault, by M. Rightmire
www.sciencedude.ocregister.com



[1] J.D. Garaud, L. Fleitout, G. Cailletaud Colloque CSMA (2009)

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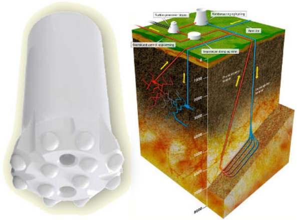
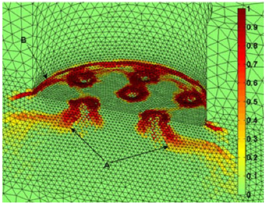
12. Deep drilling



Drill Bit tool RobitRocktools*;*
*extraction of geothermal energy (*SINTEF, NTNU*)*



[1] T. Saksala, Int. J. Numer. Anal. Meth. Geomech. (2012)

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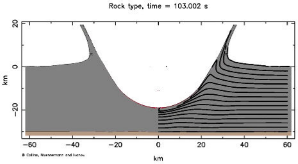
10. Electric contacts
11. Tectonic motions
12. Deep drilling
13. Impact and fragmentation

Impact crater, Arizona
www.MrEclipse.com *et* maps.google.com



Simulation of formation of Copernicus crater
Yue Z., Johnson B. C., et al. Projectile remnants in central peaks of lunar impact craters. Nature Geo 6 (2013)

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14. etc.

Impact crater, Arizona
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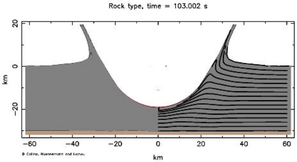

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Physical and mathematical complexity

- Contact interface is hard to observe in situ
- Many things happen in the interface
- Strong thermo-mechanical or fluid-solid coupling in sliding
- Mathematical formulation is also non-trivial, hard to handle analytically
- **Robust and accurate computational framework is needed**

Basics of Contact and Friction

Notations

Vectors and tensors

• a, α	scalars	• $\underline{a} \cdot \underline{b} = c$	scalar (dot) product
• $\underline{b}$	vectors	• $\underline{a} \times \underline{b} = \underline{c}$	vector (cross) product
• $\underline{\underline{C}}, \underline{\underline{\beta}}$	2nd order tensors	• $\underline{a} \otimes \underline{b} = \underline{\underline{C}}$	tensor product
• $^4\underline{\underline{D}}$	4th order tensors	• $\underline{\underline{A}}^T$	transposition
• $\nabla \underline{a} = \underline{\underline{B}}$	gradient operator	• $\nabla \cdot \underline{a} = c$	divergence operator
• $\nabla \times \underline{a} = \underline{b}$	curl (rot) operator	• $\underline{\underline{I}} = \underline{e}_i \otimes \underline{e}_i$	2nd order identity tensor

Mechanics

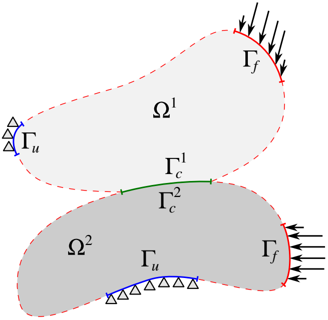
• $\underline{\underline{\sigma}}$	Cauchy stress tensor	• $\underline{\underline{\varepsilon}}$	Small strain tensor
• g, g_n	gap, normal gap	• $\underset{\sim}{\xi}$	position vector in parent space
• ϵ	penalty parameter	• $\underline{n}$	outward unit normal vector
• $\lambda, \lambda_n, \lambda_t$	lagrange multipliers	• $\frac{\partial \underline{\rho}}{\partial \xi_1}, \frac{\partial \underline{\rho}}{\partial \xi_2}$	surface tangent vectors
• $\sigma_n = (\underline{\underline{\sigma}} \cdot \underline{n}) \cdot \underline{n}$	contact pressure	• f, μ	Coefficient of friction

- **Balance of momentum**

$$\begin{cases} \nabla \cdot \underline{\underline{\sigma}} + \underline{f_v} = 0 & \text{in } \Omega_{1,2} \\ \underline{\underline{\sigma}} \cdot \underline{n} = \underline{t}_0 & \text{on } \Gamma_f \\ \underline{u} = \underline{u}_0 & \text{on } \Gamma_u \\ \;\textbf{?} & \text{on } \Gamma_c \end{cases}$$

- **Frictionless contact conditions *(intuitive)***

1. No penetration
2. No adhesion
3. No shear transfer



Equilibrium and contact conditions

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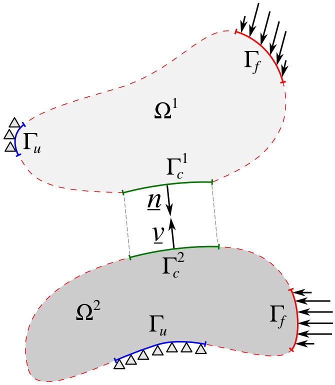

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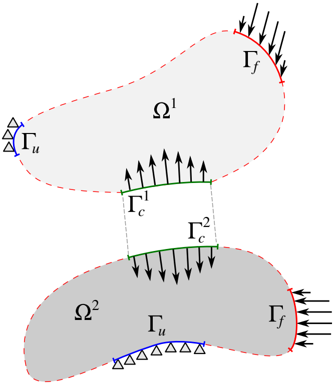

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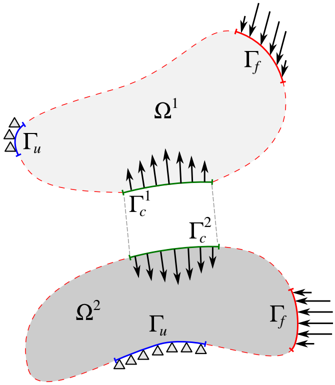

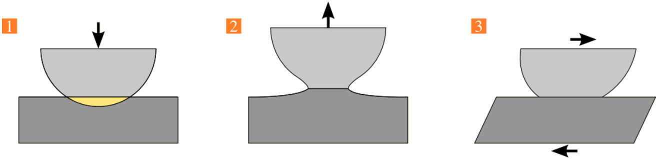

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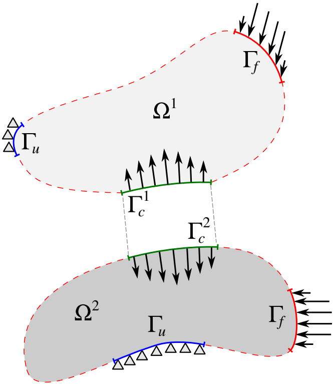
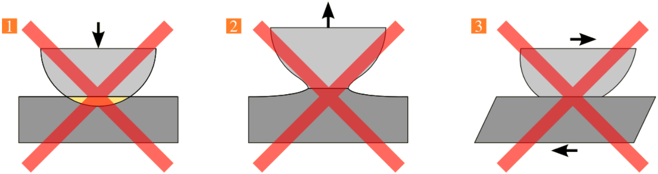
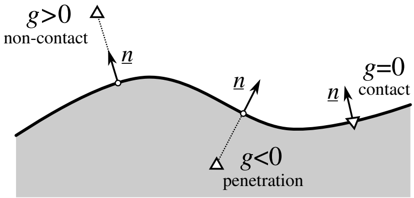




- **Gap function** g
  - gap = – penetration
  - asymmetric function
  - defined for
    - separation $g > 0$
    - contact $g = 0$
    - penetration $g < 0$
  - governs normal contact
- **Master and slave split**

  Gap function is determined for all slave points with respect to the master surface



Gap between a slave point and a master surface

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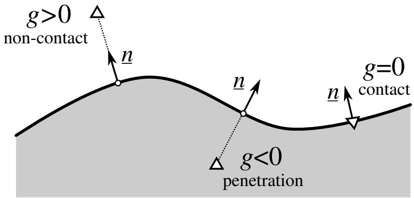
- **Master and slave split**

  Gap function is determined for all slave points with respect to the master surface
- **Normal gap**

$$g_n = \underline{n} \cdot \left[\underline{r}_s - \underline{\rho}(\xi_\pi)\right],$$

$\underline{n}$ is a unit normal vector, $\underline{r}_s$ slave point, $\underline{\rho}(\xi_\pi)$ projection point at master surface



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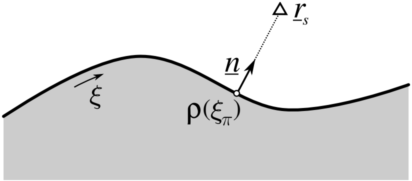


Definition of the normal gap

Consider existence and uniqueness

Frictionless or normal contact conditions

- **No penetration**
  Always non-negative gap

  $$g \geq 0$$

- **No adhesion**
  Always non-positive contact pressure

  $$\sigma_n^* \leq 0$$

- **Complementary condition**
  Either zero gap and non-zero pressure, or non-zero gap and zero pressure

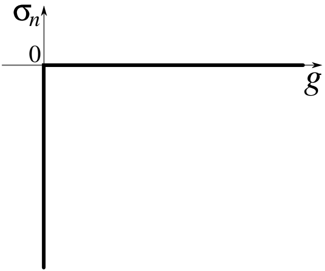
  $$g\,\sigma_n = 0$$

- **No shear transfer** *(automatically)*

  $$\underline{\sigma}_t^{**} = 0$$

$\sigma_n^* = (\underline{\underline{\sigma}} \cdot \underline{n}) \cdot \underline{n} = \underline{\underline{\sigma}} : (\underline{n} \otimes \underline{n})$

$\underline{\sigma}_t^{**} = \underline{\underline{\sigma}} \cdot \underline{n} - \sigma_n \underline{n} = \underline{n} \cdot \underline{\underline{\sigma}} \cdot \left(\underline{\underline{I}} - \underline{n} \otimes \underline{n}\right)$



Scheme explaining normal contact conditions

Frictionless or normal contact conditions

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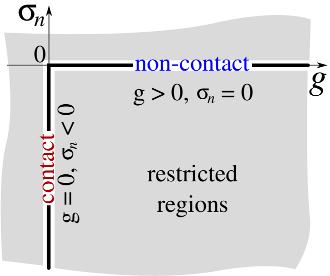


Improved scheme explaining normal contact conditions

Frictionless or normal contact conditions

In **mechanics:**

Normal contact conditions

≡

Frictionless contact conditions

≡

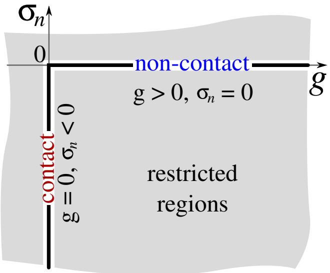
Hertz[1]*-Signorini*[2] *conditions*

≡

Hertz[1]*-Signorini*[2]*-Moreau*[3] *conditions*

also known in **optimization theory** as

Karush[4]*-Kuhn*[5]*-Tucker*[6] *conditions*



Improved scheme explaining normal contact conditions

$$g \geq 0, \qquad \sigma_n \leq 0, \qquad g\sigma_n = 0$$

[1] **Heinrich Rudolf Hertz** (1857–1894) a German physicist who first formulated and solved the frictionless contact problem between elastic ellipsoidal bodies.

[2] **Antonio Signorini** (1888–1963) an Italian mathematical physicist who gave a general and rigorous mathematical formulation of contact constraints.

[3] **Jean Jacques Moreau** (1923) a French mathematician who formulated a non-convex optimization problem based on these conditions and introduced pseudo-potentials in contact mechanics.

[4] **William Karush** (1917–1997), [5] **Harold William Kuhn** (1925) American mathematicians,

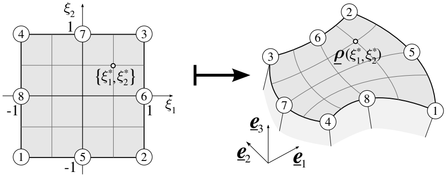
[6] **Albert William Tucker** (1905–1995) a Canadian mathematician.

Relative sliding

Recall:

- Convective coordinate in parent space $\xi_i \in (-1;1)$
- Mapping to real space

$$\underline{\rho}(\xi_1,\xi_2,t) = \sum_{i=1}^{8} N^i(\xi_1,\xi_2)\underline{\rho}^i(t)$$



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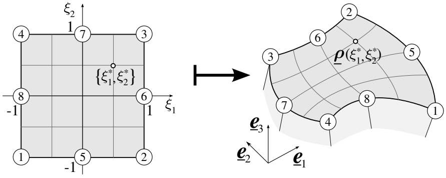
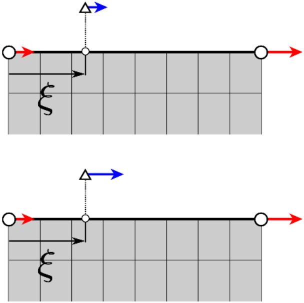


- **Tangential slip velocity** $\underline{v}_t$ must take into account:
  - only tangential component
  - relative rigid body motion
  - master's deformation

$$\underline{v}_t = \frac{\partial \underline{\rho}}{\partial \xi_1}\dot{\xi}_1 + \frac{\partial \underline{\rho}}{\partial \xi_2}\dot{\xi}_2$$

where $\partial\underline{\rho}/\partial\xi_i$ are the tangent vectors of the local basis and ξ_i are the convective coordinates.



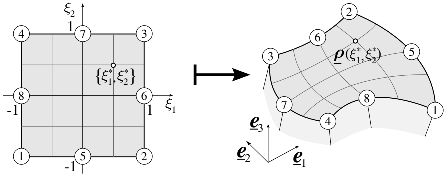
Relative slip between a slave point and a deformable master surface

Relative sliding

Recall:

- Convective coordinate in parent space $\xi_i \in (-1;1)$
- Mapping to real space

$$\underline{\rho}(\xi_1,\xi_2,t) = \sum_{i=1}^{8} N^i(\xi_1,\xi_2)\underline{\rho}^i(t)$$



- **Tangential slip velocity $\underline{v}_t$** must take into account:
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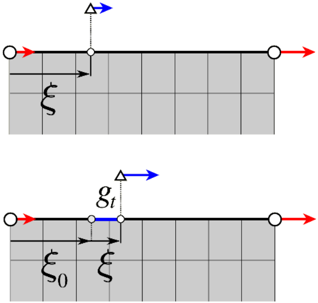

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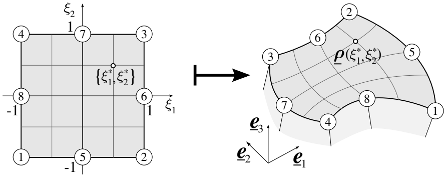

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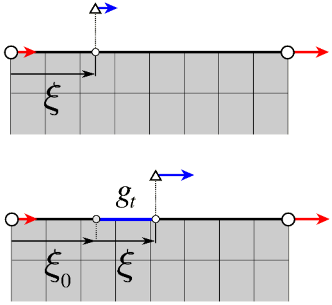

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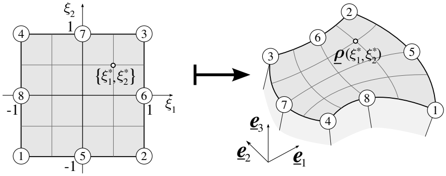

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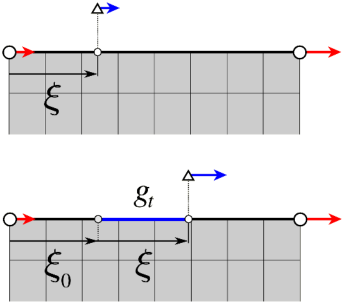

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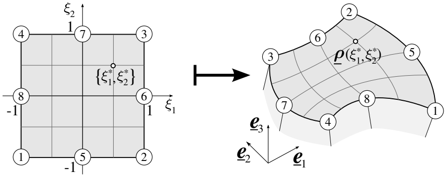

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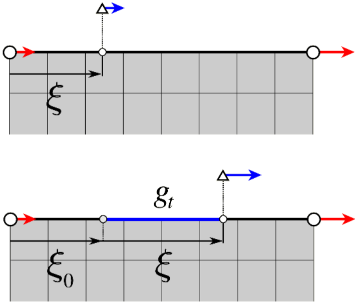

Relative slip between a slave point and a deformable master surface

Relative sliding

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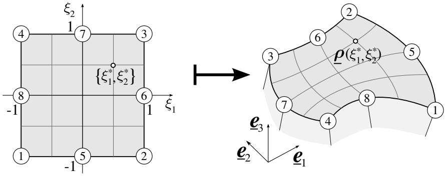

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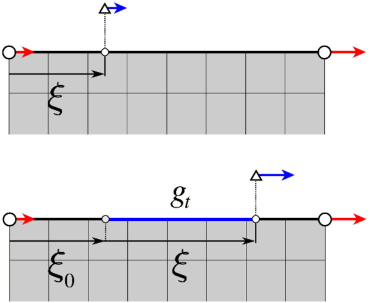


Relative slip between a slave point and a deformable master surface

Relative sliding: example

Consider a one-dimensional example:
P is a projection of A on segment BC.

$$x_P = \xi x_C + (1-\xi)x_B \qquad (1)$$

Velocity of the projection point

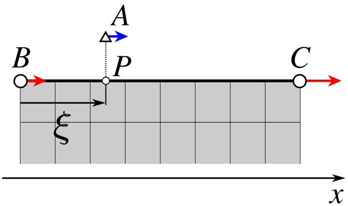
$$\dot x_P = \underbrace{\xi \dot x_C + (1-\xi)\dot x_B}_{\frac{\partial x_P}{\partial t}} + \underbrace{(x_C - x_B)\dot\xi}_{\frac{\partial x_P}{\partial \xi}\dot\xi}$$

Substract the velocity of point x_P for fixed ξ

$$v_t = \dot x_P - \frac{\partial x_P}{\partial t} = (x_C - x_B)\dot\xi = \frac{\partial x}{\partial \xi}\dot\xi$$

Compute tangential slip increment

$$\Delta g_t^{n+1} = \left.\frac{\partial x}{\partial \xi}\right|_{\xi^n} (\xi^{n+1} - \xi^n)$$



Example of a one-dimensional relative slip

Relative sliding: example

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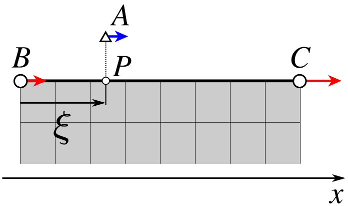
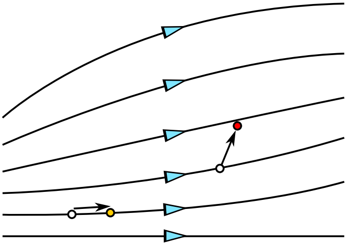


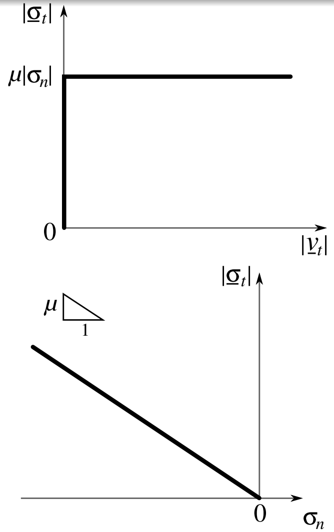
Example of a one-dimensional relative slip



Fisherman's analogy: observing sea flow around the boat.
Lie derivative: the change of a vector field along the change of another vector field

Amontons-Coulomb's friction

- **No contact** $g > 0, \ \sigma_n = 0$
- **Stick** $|\underline{v}_t| = 0$
  Inside slip surface/Coulomb's cone
  $$f = |\underline{\sigma}_t| - \mu|\sigma_n| < 0$$
- **Slip** $|\underline{v}_t| > 0$
  On slip surface/Coulomb's cone
  $$f = |\underline{\sigma}_t| - \mu|\sigma_n| = 0$$
- **Complementary condition**
  One is zero another one is not or vice versa
  $$|\underline{v}_t|\left(|\underline{\sigma}_t| - \mu|\sigma_n|\right) = 0$$
- **Direction of friction**
  Shear and sliding are collinear
  $$\underline{v}_t \parallel \underline{\sigma}_t$$



Scheme explaining frictional contact conditions

- **No contact** $g > 0,\ \sigma_n = 0$
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  Inside slip surface/Coulomb's cone
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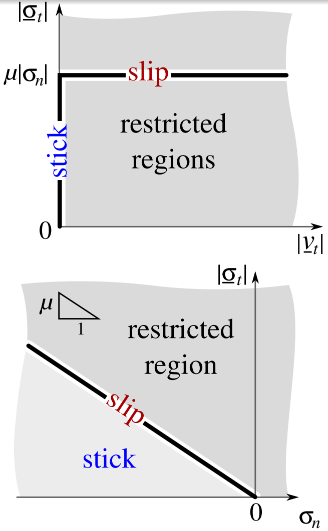

Improved scheme explaining frictional contact conditions

- **No contact** $g > 0,\ \sigma_n = 0$
- **Stick** $|\underline{v}_t| = 0$
  Inside slip surface/Coulomb's cone
  $$f = |\underline{\sigma}_t| - \mu|\sigma_n| < 0$$
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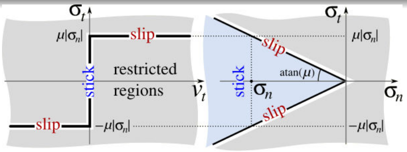
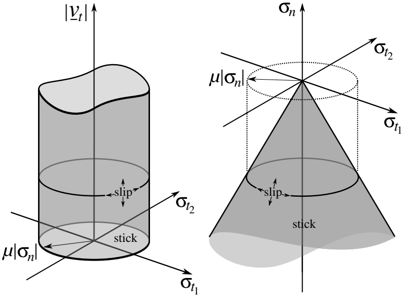


Scheme of 2D frictional contact

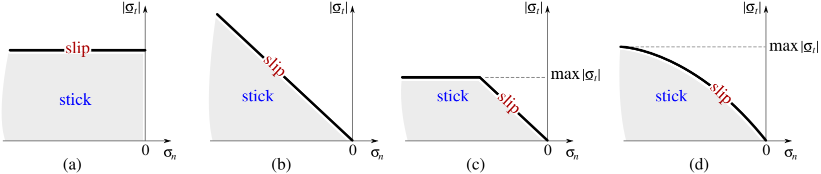


Scheme of 3D frictional contact

$$|\underline{v}_t| \ge 0, \quad |\underline{\sigma}_t| - \mu|\sigma_n| \le 0, \quad |\underline{v}_t|\left(|\underline{\sigma}_t| - \mu|\sigma_n|\right) = 0, \quad \frac{\underline{\sigma}_t}{|\underline{\sigma}_t|} = -\frac{\underline{v}_t}{|\underline{v}_t|}$$

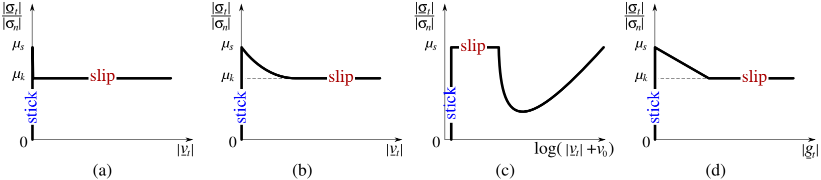
More friction laws

- **Static criteria**



(a) Tresca (b) Amontons-Coulomb (c) Coulomb-Orowan (d) Shaw

- **Kinetic criteria**



(a,b) velocity weakening (c) velocity weakening-strengthening
(d) Linear slip weakening

- μ_s static and μ_k kinetic coefficients of friction.

Rate and state friction and regularization

- **Rate and state friction law**
  - Rate $v_t = |\underline{v}_t|$ – relative slip velocity
  - State θ – $\approx$ internal time
  - Dieterich–Ruina–Perrin (1979, 83, 95)

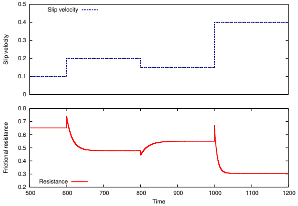
    Frictional resistance

    $$\sigma_t^c = |\sigma_n| \left[\mu_s + b\theta + a\ln(v_t/v_0)\right]$$

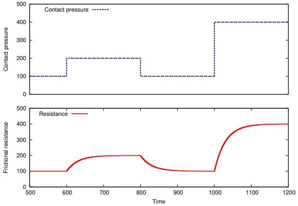
    Evolution of the state variable

    $$\dot\theta = -\frac{v_t}{L}\left[\theta + \ln\left(\frac{v_t}{v_0}\right)\right]$$

- **Prakash-Clifton friction law (1992,2000)**
  - Viscous type evolution of frictional resistance σ_t
  - $\dot\sigma_t = -\frac{v_t}{L}(\sigma_t + \mu\sigma_n)$



Rate and state friction law



Prakash-Clifton regularization

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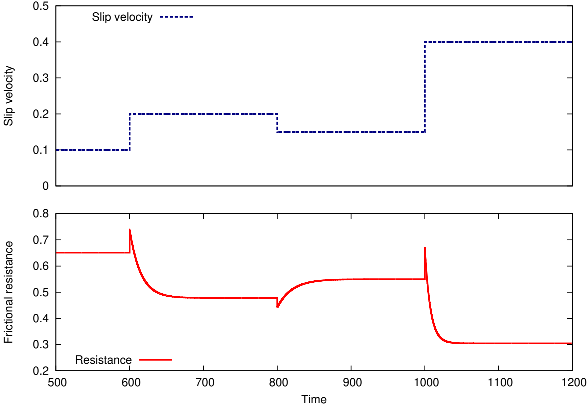

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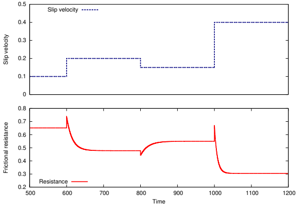

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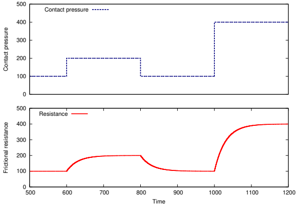

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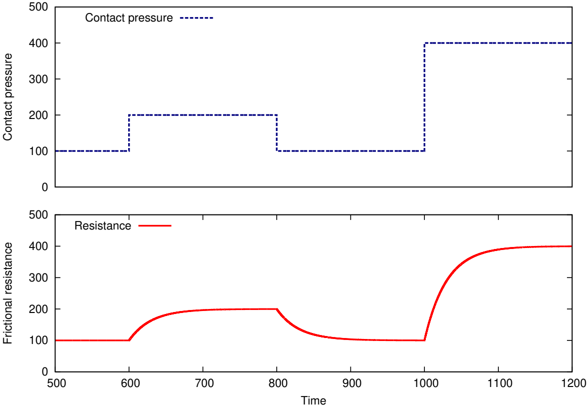

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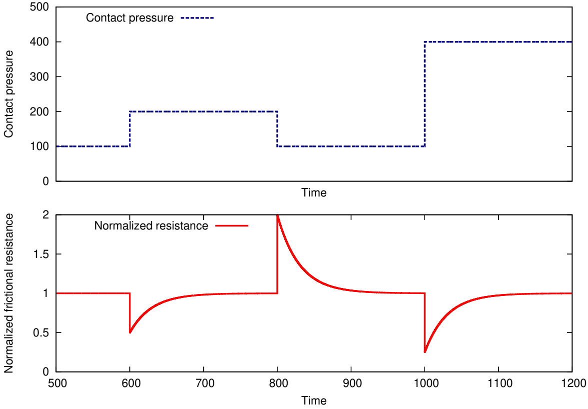
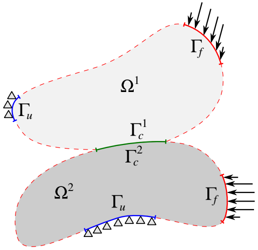


Towards a weak form

From strong to a weak form

- Balance of momentum and boundary conditions

$$\nabla \cdot \underline{\underline{\sigma}} + \underline{f}_v = 0 \quad \text{in } \Omega = \Omega_1 \cup \Omega_2 \qquad + \text{B.C.}$$



Two solids in contact

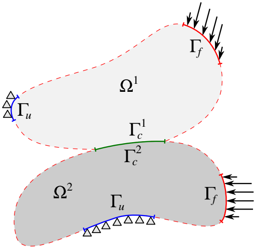
From strong to a weak form

- Balance of momentum and boundary conditions

$$\nabla\cdot\underline{\underline{\sigma}} + \underline{f}_v = 0 \quad \text{in } \Omega = \Omega_1 \cup \Omega_2 \qquad + \text{B.C.}$$

- Balance of virtual works 

$$\boxed{\int\limits_{\partial\Omega} \underline{n}\cdot\underline{\underline{\sigma}}\cdot\delta\underline{u}\,d\Gamma} + \int\limits_{\Omega} \left[\underline{f}_v\cdot\delta\underline{u} - \underline{\underline{\sigma}}\cdot\cdot\,\delta\nabla\underline{u}\right] d\Omega = 0$$



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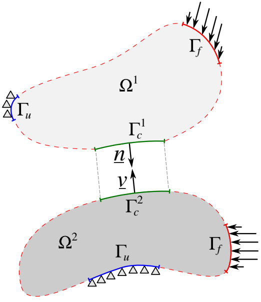
$$\boxed{\int\limits_{\partial\Omega} \underline{n} \cdot \underline{\underline{\sigma}} \cdot \delta\underline{u}\, d\Gamma} =$$

$$\boxed{\int\limits_{\bar\Gamma_c^1} \underline{n} \cdot \underline{\underline{\sigma}} \cdot \delta\underline{\rho}\, d\bar\Gamma_c^1 + \int\limits_{\bar\Gamma_c^2} [\cdot]\underline{\underline{\sigma}} \cdot \delta\underline{r}\, d\bar\Gamma_c^2} + \int\limits_{\Gamma_f} \underline{\sigma}_0 \cdot \delta\underline{u}\, d\Gamma_f$$



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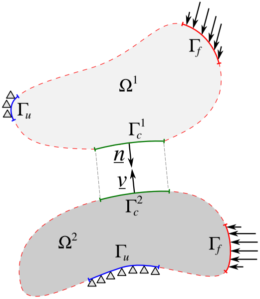
$$\boxed{\int\limits_{\partial\Omega} \underline{n} \cdot \underline{\underline{\sigma}} \cdot \delta \underline{u} \, d\Gamma} \Rightarrow$$

$$\boxed{\int\limits_{\bar\Gamma_c^1} \underline{n} \cdot \underline{\underline{\sigma}} \cdot \delta \underline{\rho} \, d\bar\Gamma_c^1 + \int\limits_{\bar\Gamma_c^2} [\cdot] \underline{\underline{\sigma}} \cdot \delta \underline{r} \, d\bar\Gamma_c^2} =$$

$$= \int\limits_{\bar\Gamma_c^1} \underline{n} \cdot \underline{\underline{\sigma}} \cdot \delta(\underline{\rho} - \underline{r}) \, d\bar\Gamma_c^1 = \int\limits_{\bar\Gamma_c^1} \left(\sigma_n \delta g_n + \underset{\sim}{\sigma}{}_t^T \delta \underset{\sim}{\xi} \right) d\bar\Gamma_c^1$$



Two solids in contact

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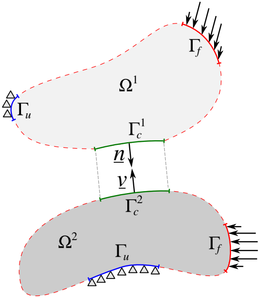
$$\int\limits_{\Omega}\underline{\underline{\sigma}}\cdot\cdot\delta\nabla\underline{u}\,d\Omega + \underbrace{\boxed{\int\limits_{\overline{\Gamma}_c^1}\left(\sigma_n\delta g_n + \underset{\sim}{\sigma}{}_t^T\delta\underset{\sim}{\xi}\right)d\overline{\Gamma}_c^1}}_{\text{Contact term}} = \int\limits_{\Gamma_f}\underline{\sigma}_0\cdot\delta\underline{u}\,d\Gamma + \int\limits_{\Omega}\underline{f}_v\cdot\delta\underline{u}\,d\Omega$$



Two solids in contact

From strong to a weak form

- Balance of momentum and boundary conditions

$$\nabla \cdot \underline{\underline{\sigma}} + \underline{f}_v = 0 \quad \text{in } \Omega = \Omega_1 \cup \Omega_2 \qquad + \text{B.C.}$$

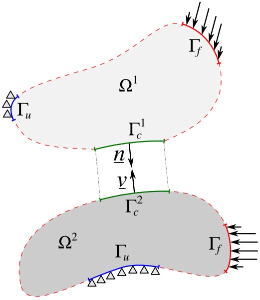
- Balance of virtual works 

$$\underbrace{\int\limits_{\Omega} \underline{\underline{\sigma}} \cdot\cdot \delta\nabla\underline{u}\, d\Omega}_{\text{Change of the internal energy}} + \underbrace{\boxed{\int\limits_{\bar\Gamma_c^1} \left(\sigma_n \delta g_n + \underset{\sim}{\sigma}{}_t^T \delta \underset{\sim}{\xi}\right) d\bar\Gamma_c^1}}_{\text{Contact term}} =$$

$$\underbrace{\int\limits_{\Gamma_f} \underline{\sigma}_0 \cdot \delta\underline{u}\, d\Gamma}_{\text{Virtual work of external forces}} + \underbrace{\int\limits_{\Omega} \underline{f}_v \cdot \delta\underline{u}\, d\Omega}_{\text{Virtual work of volume forces}}$$

- Functional **space**

$\delta\underline{u}, \underline{u} \in \mathbb{H}^1(\Omega)$ Hilbert space of the first order (first derivate is square integrable) and $\delta\underline{u}, \underline{u}$ satisfy boundary conditions



Two solids in contact

From strong to a weak form

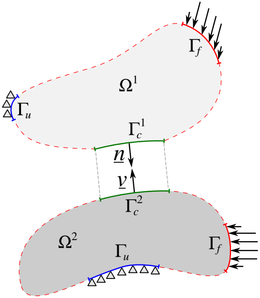
- Balance of momentum and boundary conditions

$$\nabla \cdot \underline{\underline{\sigma}} + \underline{f}_v = 0 \quad \text{in } \Omega = \Omega_1 \cup \Omega_2 \qquad + \text{B.C.}$$

- Balance of virtual works 

$$\underbrace{\int\limits_{\Omega} \underline{\underline{\sigma}} \cdot\cdot \delta \nabla \underline{u} \, d\Omega}_{\text{Change of the internal energy}} + \underbrace{\boxed{\int\limits_{\bar{\Gamma}_c^1} \left(\sigma_n \delta g_n + \underset{\sim}{\sigma}_t^T \delta \underset{\sim}{\xi} \right) d\bar{\Gamma}_c^1}}_{\text{Contact term}} \geq$$

$$\underbrace{\int\limits_{\Gamma_f} \underline{\sigma}_0 \cdot \delta \underline{u} \, d\Gamma}_{\text{Virtual work of external forces}} + \underbrace{\int\limits_{\Omega} \underline{f}_v \cdot \delta \underline{u} \, d\Omega}_{\text{Virtual work of volume forces}}$$



Two solids in contact

- Functional **subspace**

$\delta \underline{u}, \underline{u} \in \mathbb{H}^1(\Omega)$ Hilbert space of the first order (first derivate is square integrable) and $\delta \underline{u}, \underline{u}$ satisfy boundary conditions and **contact conditions**, so we do optimization on a (potentially nonconvex) subset of $\mathbb{H}^1(\Omega)$.

Variational inequality

- Optimization problem for $F : \mathbb{V} \to \mathbb{R}$
- Find $u \in \mathbb{V}$ s.t. $\forall v \in \mathbb{V} : \; F(u) \le F(v)$
- If $F \in C^1$ is convex then such minimizer u is a stationary point $F'|_u = 0$

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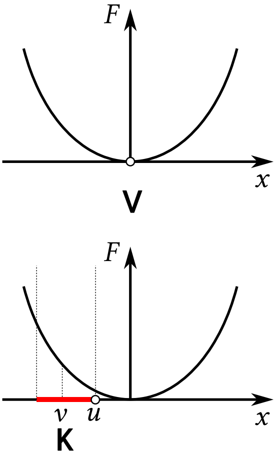
- Variational inequality for minimizer $u \in \mathbb{K} \subset \mathbb{V}$:

$$\boxed{F'(u)(v-u) \ge 0, \quad \forall v \in \mathbb{K}}$$

Example of variational inequality



Minimize $F(x)$ for $x \in \mathbb{K} \subset \mathbb{R}$, then the minimizer u satisfies

$$F'(u)(v-u) \geq 0, \quad \forall v \in \mathbb{K}$$

Variational inequality in contact

- Since $g_n\sigma_n = 0$, then $\sigma_n\delta g_n + \delta\sigma_n g_n = 0$
- The corresponding variational inequality:

$$\int\limits_{\Omega} \underline{\underline{\sigma}} \cdot\cdot \delta\nabla\underline{u}\, d\Omega + \int\limits_{\bar\Gamma_c^1} \underset{\sim}{\sigma}_t^T \delta\underset{\sim}{\xi}\, d\bar\Gamma_c^1 \ge \int\limits_{\Gamma_f} \underline{\sigma}_0 \cdot \delta\underline{u}\, d\Gamma + \int\limits_{\Omega} \underline{f}_v \cdot \delta\underline{u}\, d\Omega, \quad \underline{u} \in \mathbb{L}, \delta\underline{u} \in \mathbb{K}$$

$$\mathbb{L} = \left\{ \underline{u} \in \mathbb{H}^1(\Omega) \;\middle|\; \underline{u} = \underline{u}_0 \text{ on } \Gamma_u,\; g_n(\underline{u}) \ge 0 \text{ on } \Gamma_c \right\}$$
$$\mathbb{K} = \left\{ \delta\underline{u} \in \mathbb{H}^1(\Omega) \;\middle|\; \delta\underline{u} = 0 \text{ on } \Gamma_u,\; g_n(\underline{u} + \delta\underline{u}) \ge 0 \text{ on } \Gamma_c \right\}$$

Back to variational equality (unconstrained)

- Constrained minimization problem

$$\int\limits_{\Omega} \underline{\underline{\sigma}} \cdot\cdot \delta \nabla \underline{u} \, d\Omega + \int\limits_{\bar{\Gamma}_c^1} \underset{\sim}{\sigma}_t^T \delta \underset{\sim}{\xi} \, d\bar{\Gamma}_c^1 \geq \int\limits_{\Gamma_f} \underline{\sigma}_0 \cdot \delta \underline{u} \, d\Gamma + \int\limits_{\Omega} \underline{f}_v \cdot \delta \underline{u} \, d\Omega, \quad \underline{u} \in \mathbb{L}, \delta \underline{u} \in \mathbb{K}$$

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- Use optimization theory to convert to

$$\int\limits_{\Omega} \underline{\underline{\sigma}} \cdot\cdot \delta\nabla\underline{u}\, d\Omega + \int\limits_{\Gamma_c^1} \underbrace{\boxed{C(\sigma_n, \sigma_t, g_n, \underset{\sim}{\xi}, \delta\underline{u})}}_{\text{Contact term}^*} d\Gamma_c^1 = \int\limits_{\Gamma_f} \underline{\sigma}_0 \cdot \delta\underline{u}\, d\Gamma + \int\limits_{\Omega} \underline{f}_v \cdot \delta\underline{u}\, d\Omega,$$

Unconstrained functional sub-spaces

$$\mathbb{L} = \left\{\underline{u} \in \mathbb{H}^1(\Omega) \;\middle|\; \underline{u} = \underline{u}_0 \text{ on } \Gamma_u\right\}$$

$$\mathbb{K} = \left\{\delta\underline{u} \in \mathbb{H}^1(\Omega) \;\middle|\; \delta\underline{u} = 0 \text{ on } \Gamma_u\right\}$$

Contact term* is defined on the *potential contact zone* Γ_c^1.

Optimization methods

Optimization methods: recall

Functional to be minimized $F(\mathbf{x})$ under constraint $g(\mathbf{x}) \ge 0$

- Penalty method
- Lagrange multipliers method
- Augmented Lagrangian method

Optimization methods: recall

Functional to be minimized $F(\mathbf{x})$ under constraint $g(\mathbf{x}) \geq 0$

- **Penalty method**
  - New functional

$$F_p(\mathbf{x}) = F(\mathbf{x}) + \boxed{\epsilon \langle -g(\mathbf{x}) \rangle^2} = F(\mathbf{x}) + \begin{cases} 0, & \text{if } g(\mathbf{x}) \geq 0 \quad \textit{non-contact} \\ \epsilon g^2(\mathbf{x}), & \text{if } g(\mathbf{x}) < 0 \quad \textit{contact} \end{cases}$$

  where ϵ is the penalty parameter.
  - Stationary point must satisfy

$$\nabla F_p(\mathbf{x}) = \nabla F(\mathbf{x}) + 2\epsilon \langle -g(\mathbf{x}) \rangle \nabla g(\mathbf{x}) = 0$$

  - Solution **tends** to the precise solution as $\epsilon \to \infty$
- Lagrange multipliers method
- Augmented Lagrangian method

Macaulay brackets $\langle x \rangle = \begin{cases} x, & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases}$

Optimization methods: recall

Functional to be minimized $F(\mathbf{x})$ under constraint $g(\mathbf{x}) \geq 0$

- Penalty method $F_p(\mathbf{x}) = F(\mathbf{x}) + \boxed{\epsilon\left\langle -g(\mathbf{x})\right\rangle^2}$
- **Lagrange multipliers method**
  - New functional called **Lagrangian**

$$\mathcal{L}(\mathbf{x},\lambda) = F(\mathbf{x}) + \boxed{\lambda g(\mathbf{x})}$$

  - Saddle point problem

$$\min_{x}\max_{\lambda}\{\mathcal{L}(\mathbf{x},\boldsymbol{\lambda})\} \longrightarrow \mathbf{x}^* \longleftarrow \min_{g(\mathbf{x})\geq 0}\{F(\mathbf{x})\}$$

  - Stationary point

$$\nabla_{\mathbf{x},\lambda}\mathcal{L} = \begin{bmatrix}\nabla_{\mathbf{x}}F(\mathbf{x}) + \lambda\nabla_{x}g(\mathbf{x}) \\ g(\mathbf{x})\end{bmatrix} = 0 \quad \text{need to verify} \quad \lambda \leq 0$$

- Augmented Lagrangian method

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Functional to be minimized $F(\mathbf{x})$ under constraint $g(\mathbf{x}) \geq 0$

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- Lagrange multipliers method $\mathcal{L}(\mathbf{x}, \lambda) = F(\mathbf{x}) + \boxed{\lambda g(\mathbf{x})}$
- **Augmented Lagrangian method**
  [Hestnes 1969], [Powell 1969], [Glowinski & Le Tallec 1989], [Alart & Curnier 1991], [Simo & Laursen 1992]
  - New functional, augmented Lagrangian

$$\mathcal{L}_a(\mathbf{x}, \lambda) = F(\mathbf{x}) + \begin{cases} \boxed{\lambda g(\mathbf{x})} + \boxed{\epsilon g^2(\mathbf{x})}, & \text{if } \lambda + 2\epsilon g(\mathbf{x}) \leq 0, \; \textit{contact} \\ -\frac{1}{4\epsilon}\lambda^2, & \text{if } \lambda + 2\epsilon g(\mathbf{x}) > 0, \; \textit{non-contact} \end{cases}$$

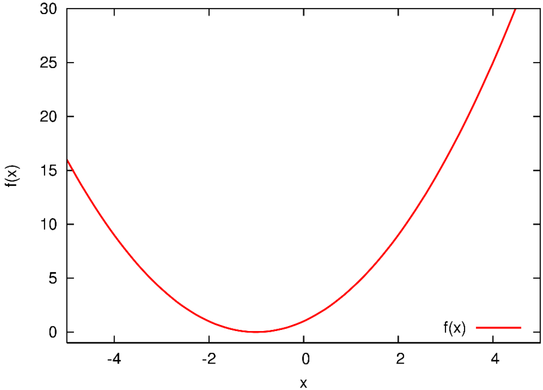
  - Stationary point

$$\nabla_{\mathbf{x},\lambda} \mathcal{L}_a = \begin{cases} \begin{bmatrix} \nabla_{\mathbf{x}} F(\mathbf{x}) + \lambda \nabla_x g(\mathbf{x}) + 2\epsilon g(\mathbf{x}) \nabla g(\mathbf{x}) \\ g(\mathbf{x}) \end{bmatrix} = 0, & \textit{if contact} \\ \begin{bmatrix} \nabla_{\mathbf{x}} F(\mathbf{x}) \\ -\frac{\lambda}{\epsilon} \end{bmatrix} = 0, & \textit{if non-contact} \end{cases}$$

Macaulay brackets $\langle x \rangle = \begin{cases} x, & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases}$


Uzawa algorithm

Optimization methods: example

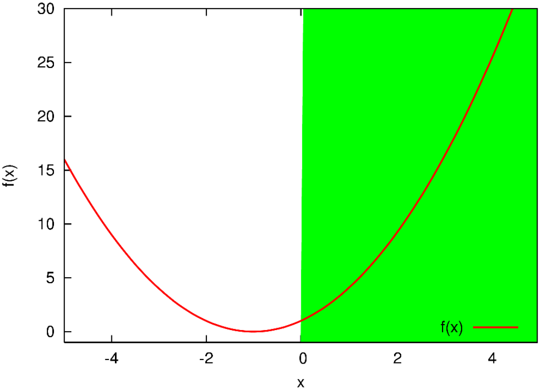


Functional : $f(x) = x^2 + 2x + 1$

Constrain : $g(x) = x \geq 0$

Solution : $x^* = 0$

Optimization methods: example



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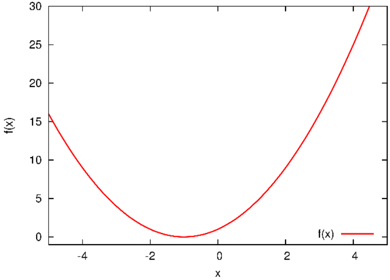
Penalty method: example

$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

- **Penalty method**

$$F_p(x) = F(x) + \boxed{\epsilon \left\langle -g(x) \right\rangle^2}$$



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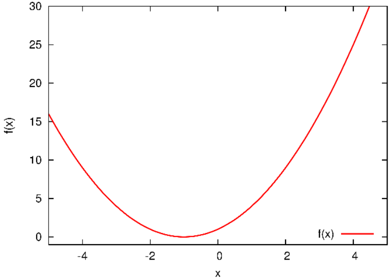

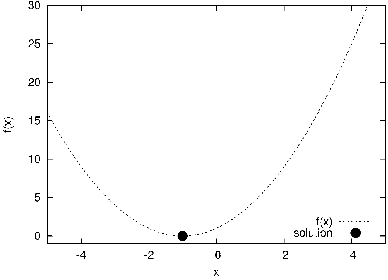

$\epsilon = 0$

Penalty method: example

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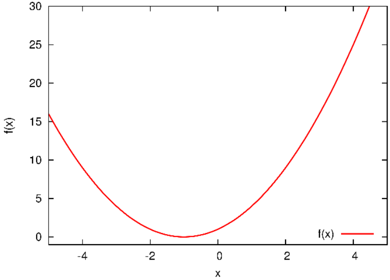
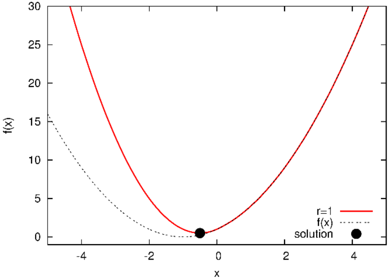
- **Penalty method**

$$F_p(x) = F(x) + \boxed{\epsilon \langle -g(x) \rangle^2}$$





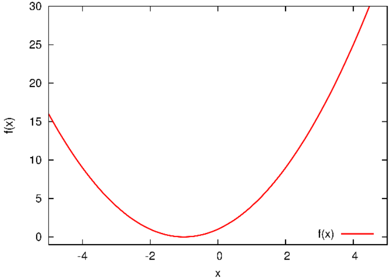
$\epsilon = 1$

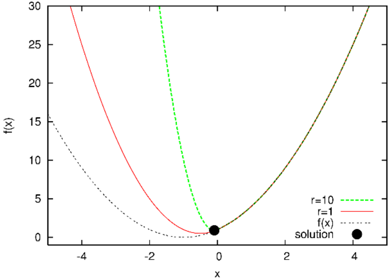
Penalty method: example

$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

- **Penalty method**

$$F_p(x) = F(x) + \boxed{\epsilon \langle -g(x) \rangle^2}$$





$\epsilon = 10$

Penalty method: example

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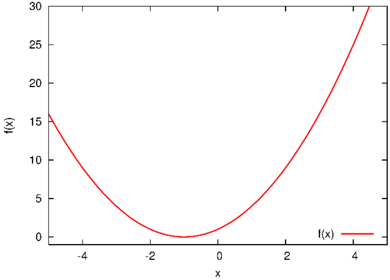
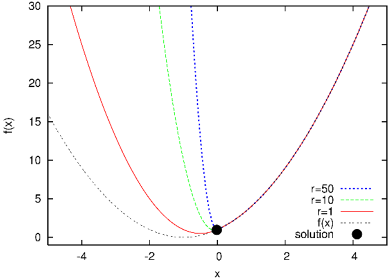




$\epsilon = 50$

Penalty method: example

$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

- **Penalty method**

$$F_p(x) = F(x) + \boxed{\epsilon \langle -g(x) \rangle^2}$$

Advantages ☺

- simple physical interpretation
- simple implementation
- no additional degrees of freedom
- "mathematically" smooth functional

Drawbacks ☹

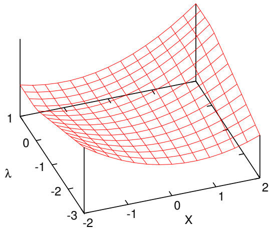
- practically non-smooth functional
- solution is not exact:
  - too small penalty → large penetration
  - too large penalty → ill-conditioning of the tangent matrix
- user has to choose penalty ϵ properly or automatically and/or adapt during convergence

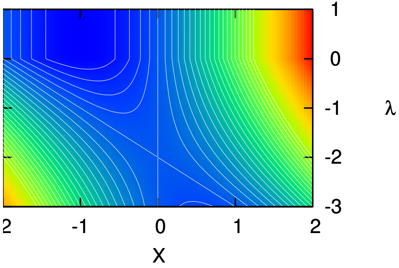
$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

- **Lagrange multipliers method**

  $\mathcal{L}(x, \lambda) = F(x) + \boxed{\lambda g(x)} \to$ Saddle point $\to \min\limits_{x} \max\limits_{\lambda} \mathcal{L}(x, \lambda)$

  Need to check that $\lambda \leq 0$





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  Need to check that $\lambda \leq 0$

Advantages ☺

- exact solution
- no adjustable parameters

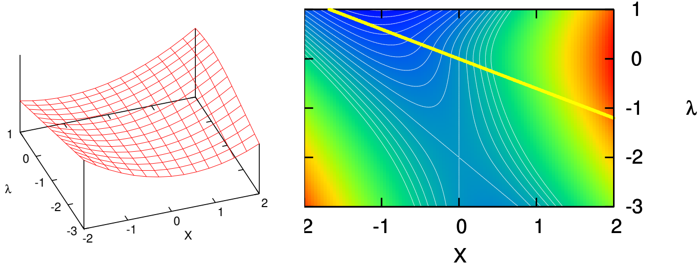
Drawbacks ☹

- Lagrangian is not smooth
- additional degrees of freedom
- not fully unconstrained: $\lambda \leq 0$

$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

Augmented Lagrangian method

$$\mathcal{L}_a(\mathbf{x}, \lambda) = F(\mathbf{x}) + \begin{cases} \boxed{\lambda g(\mathbf{x})} + \boxed{\epsilon g^2(\mathbf{x})}, & \text{if } \lambda + 2\epsilon g(\mathbf{x}) \leq 0, \textit{ contact} \\ -\frac{1}{4\epsilon}\lambda^2, & \text{if } \lambda + 2\epsilon g(\mathbf{x}) > 0, \textit{ non-contact} \end{cases}$$



Yellow line separates contact and non-contact regions

Augmented Lagrangian method: example

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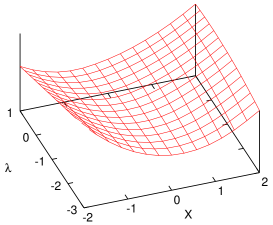

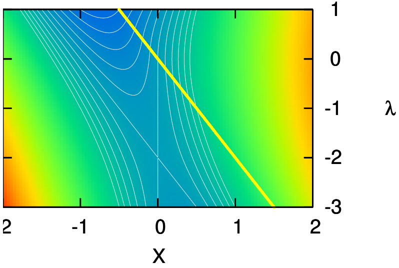

Yellow line separates contact and non-contact regions

$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

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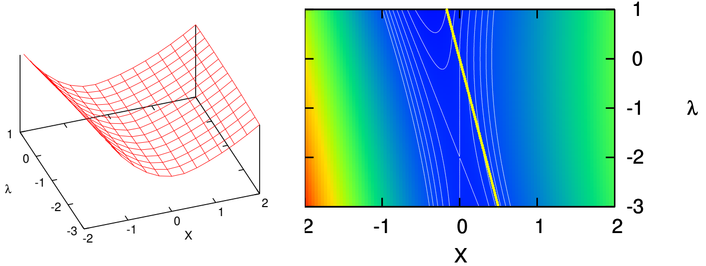

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Advantages ☺

- exact solution
- smoother functional (!)
- fully unconstrained

Drawbacks ☹

- additional degrees of freedom
- quite sensitive to parameter ϵ
- need to adjust ϵ during convergence

Augmented Lagrangian with Uzawa algorithm

Augmented Lagrangian method

$$\mathcal{L}_a(\mathbf{x},\lambda) = F(\mathbf{x}) + \begin{cases} \boxed{\lambda g(\mathbf{x})} + \boxed{\epsilon g^2(\mathbf{x})}, & \text{if } \lambda + 2\epsilon g(\mathbf{x}) \le 0,\ \textit{contact} \\ -\frac{1}{4\epsilon}\lambda^2, & \text{if } \lambda + 2\epsilon g(\mathbf{x}) > 0,\ \textit{non-contact} \end{cases}$$

Fix $\lambda = \lambda_0$

$$\mathcal{L}_a(\mathbf{x},\lambda) = F(\mathbf{x}) + \lambda_0 g(\mathbf{x}) + \epsilon g^2(\mathbf{x}),\ \text{if } \lambda_0 + 2\epsilon g(\mathbf{x}) \le 0$$

Converge with respect to x

Augmented Lagrangian with Uzawa algorithm

Augmented Lagrangian method

$$\mathcal{L}_a(\mathbf{x},\lambda) = F(\mathbf{x}) + \begin{cases} \boxed{\lambda g(\mathbf{x})} + \boxed{\epsilon g^2(\mathbf{x})}, & \text{if } \lambda + 2\epsilon g(\mathbf{x}) \le 0, \textit{ contact} \\ -\frac{1}{4\epsilon}\lambda^2, & \text{if } \lambda + 2\epsilon g(\mathbf{x}) > 0, \textit{ non-contact} \end{cases}$$

Fix $\lambda = \lambda_0$

$$\mathcal{L}_a(\mathbf{x},\lambda) = F(\mathbf{x}) + \left[\lambda_0 + \epsilon g(\mathbf{x})\right]\, g(\mathbf{x}),\ \text{if } \lambda_0 + 2\epsilon g(\mathbf{x}) \le 0$$

Converge with respect to x and update $\lambda_{i+1} = \lambda_i + \epsilon g(\mathbf{x})$

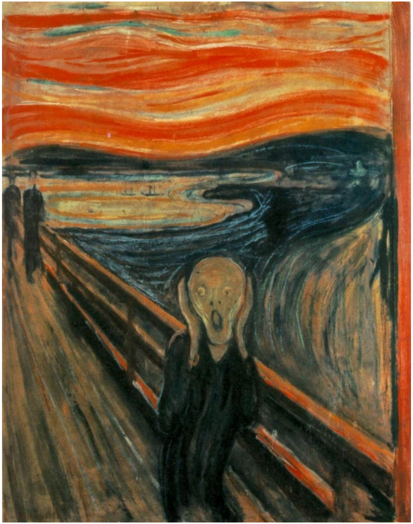
Augmented Lagrangian method

$$\mathcal{L}_a(\mathbf{x},\lambda) = F(\mathbf{x}) + \begin{cases} \boxed{\lambda g(\mathbf{x})} + \boxed{\epsilon g^2(\mathbf{x})}, & \text{if } \lambda + 2\epsilon g(\mathbf{x}) \le 0, \textit{ contact} \\ -\frac{1}{4\epsilon}\lambda^2, & \text{if } \lambda + 2\epsilon g(\mathbf{x}) > 0, \textit{ non-contact} \end{cases}$$

Fix $\lambda = \lambda_0$
Converge with respect to x and update $\lambda_{i+1} = \lambda_i + \epsilon g(\mathbf{x})$

$$\mathcal{L}_a(\mathbf{x},\lambda) = F(\mathbf{x}) + \left[\lambda_1 + \epsilon g(\mathbf{x})\right]\, g(\mathbf{x}),\ \text{if } \lambda_1 + 2\epsilon g(\mathbf{x}) \le 0$$

Friction . . . . . .



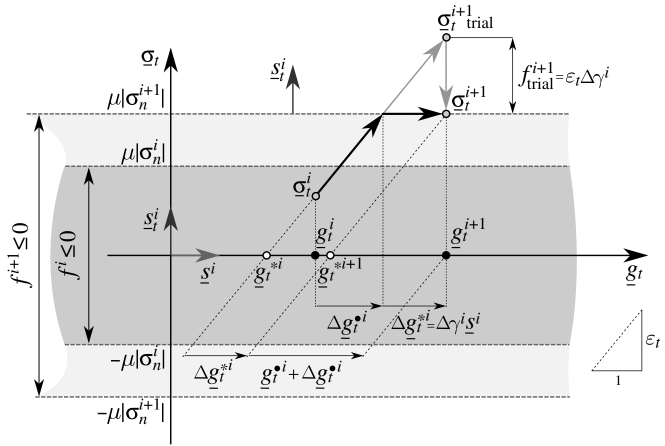
"The scream“

Friction: methods

- Optimization methods: penalty or augmented Lagrangian method
- Note that the method of Lagrange multipliers cannot be employed here
- Return mapping algorithm for penalty
- Analogy with elasto-plastic formulation problem[1]

[1] Curnier "A theory of friction" Int J Solids Struct 20 (1984)

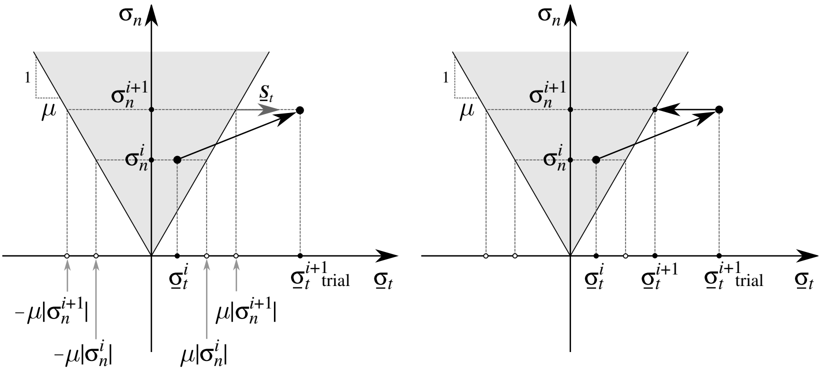
- Return mapping algorithm in 2D for the penalty method



As in plasticity[1]

[1] Simo J.C. and Hughes T.J.. Computational inelasticity. Springer (2006)

- Return mapping algorithm in 2D for the penalty method



Analogy with non-associated plastic flow[2]

[2] Curnier A. A theory of friction. International Journal of Solids and Structures 20 (1984)

Application to contact problems: weak form

$$\int\limits_{\Omega} \underline{\underline{\sigma}} \cdot\cdot \delta \nabla \underline{u} \, d\Omega + \int\limits_{\Gamma_c^1} \underbrace{\boxed{C}}_{\text{Contact term}} d\Gamma_c^1 = \int\limits_{\Gamma_f} \underline{\sigma}_0 \cdot \delta \underline{u} \, d\Gamma + \int\limits_{\Omega} \underline{f}_v \cdot \delta \underline{u} \, d\Omega,$$

$$\underline{u} \in \mathbb{L}, \delta\underline{u} \in \mathbb{K}, \qquad \mathbb{L} = \left\{ \underline{u} \in \mathbb{H}^1(\Omega) \;\middle|\; \underline{u} = \underline{u}_0 \text{ on } \Gamma_u \right\},$$

$$\mathbb{K} = \left\{ \delta\underline{u} \in \mathbb{H}^1(\Omega) \;\middle|\; \delta\underline{u} = 0 \text{ on } \Gamma_u \right\}$$

- Penalty method

Pressure: $\sigma_n = \epsilon g_n$, Shear: $\underline{\sigma}_t = \begin{cases} \epsilon \underline{g}_t, & \textit{if stick } |\sigma_t| < \mu|\sigma_n| \\ \mu \epsilon g_n \delta \underline{g}_t / |\delta \underline{g}_t|, & \textit{if slip } |\sigma_t| = \mu|\sigma_n| \end{cases}$

Contact term

$$C = C(g_n, \underline{g}_t, \delta g_n, \delta \underline{g}_t) = \sigma_n \delta g_n + \underline{\sigma}_t \cdot \delta \underline{g}_t$$

Application to contact problems: weak form

$$\int\limits_{\Omega} \underline{\underline{\sigma}} \cdot\cdot \delta \nabla \underline{u} \, d\Omega + \int\limits_{\Gamma_c^1} \underbrace{\boxed{C}}_{\text{Contact term}} \, d\Gamma_c^1 = \int\limits_{\Gamma_f} \underline{\sigma}_0 \cdot \delta \underline{u} \, d\Gamma + \int\limits_{\Omega} \underline{f}_v \cdot \delta \underline{u} \, d\Omega,$$

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- Augmented Lagrangian method

Contact term

$$C = C(g_n, \underline{g}_t, \lambda_n, \underline{\lambda}_t, \delta g_n, \delta \underline{g}_t, \delta \lambda_n, \delta \underline{\lambda}_t)$$

$$C = \begin{cases} -\frac{1}{\epsilon}\left(\lambda_n \delta \lambda_n - \underline{\lambda}_t \cdot \delta \underline{\lambda}_t\right), & \text{if non-contact } \lambda_n + \epsilon g_n \geq 0 \\ \hat{\lambda}_n \delta g_n + g_n \delta \lambda_n + \underline{\hat{\lambda}}_t \cdot \delta \underline{g}_t + \underline{g}_t \cdot \delta \underline{\hat{\lambda}}_t, & \text{if stick } |\underline{\hat{\lambda}}_t| \leq \mu |\hat{\sigma}_n| \\ \hat{\lambda}_n \delta g_n + g_n \delta \lambda_n + \mu \hat{\sigma}_n - \mu \hat{\sigma}_n \frac{\underline{\hat{\lambda}}_t}{|\underline{\hat{\lambda}}_t|} \cdot \delta \underline{g}_t - \frac{1}{\epsilon}\left(\lambda_t + \mu \hat{\sigma}_n \frac{\underline{\hat{\lambda}}_t}{|\underline{\hat{\lambda}}_t|}\right) \cdot \delta \underline{\lambda}_t, & \text{if slip } |\underline{\hat{\lambda}}_t| \geq \mu |\hat{\sigma}_n| \end{cases}$$

where $\hat{\lambda}_n = \lambda_n + \epsilon g_n$ and $\underline{\hat{\lambda}}_t = \underline{\lambda}_t + \epsilon \underline{g}_t$.

Application to contact problems: linearization

- Non-linear equation

$$R(\underline{u},\underline{f}) = 0$$

- Contains $\delta g_n, \delta \underline{g}_t$
- Use Newton-Raphson method
- Initial state at step i

$$R(\underline{u}^i,\underline{f}^i) = 0$$

- Should be also satisfied at step $i+1$

$$R(\underline{u}^{i+1},\underline{f}^{i+1}) = R(\underline{u}^i + \delta\underline{u},\underline{f}^{i+1}) = 0$$

- Linearize

$$R(\underline{u}^i + \delta\underline{u},\underline{f}^{i+1}) = R(\underline{u}^i,\underline{f}^{i+1}) + \frac{\partial R(\underline{u})}{\partial \underline{u}}\delta\underline{u} = 0$$

- Finally

$$\delta\underline{u} = - \underbrace{\left[\frac{\partial R(\underline{u})}{\partial \underline{u}}\right]^{-1}}_{\text{contains } \Delta\delta g_n,\, \Delta\delta \underline{g}_t} R(\underline{u}^i)$$

- NB: Contact problem does not satisfy conditions of Kantorovich theorem on the convergence of Newton's method.

Normal gap

- First variation enters in the residual vector:

$$\delta g_n = \underline{n} \cdot (\delta \underline{r}_s - \delta \underline{\rho})$$

- Second variation enters in the tangent matrix:

$$\Delta\delta g_n = -\underline{n} \cdot \left(\delta \frac{\partial \underline{\rho}}{\partial \underset{\sim}{\xi}}^T \Delta \underset{\sim}{\xi} + \Delta \frac{\partial \underline{\rho}}{\partial \underset{\sim}{\xi}}^T \delta \underset{\sim}{\xi} \right) - \Delta \underset{\sim}{\xi}^T \underset{\approx}{H} \delta \underset{\sim}{\xi} + \\ + g_n \left(\Delta \underset{\sim}{\xi}^T \underset{\approx}{H} + \underline{n} \cdot \Delta \frac{\partial \underline{\rho}}{\partial \underset{\sim}{\xi}}^T \right) \underset{\approx}{\bar{A}} \left(\underline{n} \cdot \delta \frac{\partial \underline{\rho}}{\partial \underset{\sim}{\xi}} + \underset{\approx}{H} \delta \underset{\sim}{\xi} \right)$$

Convective coordinate of the projection

- First variation enters in the residual vector:

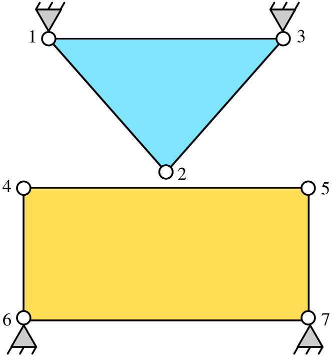
$$\delta\underset{\sim}{\xi} = \left[\underset{\approx}{A} - g_n \underset{\approx}{H}\right]^{-1}\left(\frac{\partial \underline{\rho}}{\partial \underset{\sim}{\xi}} \cdot (\delta\underline{r}_s - \delta\underline{\rho}) + g_n \underline{\boldsymbol{n}} \cdot \delta\frac{\partial \underline{\rho}}{\partial \underset{\sim}{\xi}}\right)$$

- Second variation enters in the tangent matrix:

$$\begin{aligned}\Delta\delta\underset{\sim}{\xi} = (g_n \underset{\approx}{H} - \underset{\approx}{A})^{-1}\Bigg\{ &\frac{\partial \underline{\rho}}{\partial \underset{\sim}{\xi}} \cdot \left(\delta\frac{\partial \underline{\rho}}{\partial \underset{\sim}{\xi}}^T \Delta\underset{\sim}{\xi} + \Delta\frac{\partial \underline{\rho}}{\partial \underset{\sim}{\xi}}^T \delta\underset{\sim}{\xi}\right) + \Delta\underset{\sim}{\xi}^T\left(\frac{\partial \underline{\rho}}{\partial \underset{\sim}{\xi}} \cdot \frac{\partial^2 \underline{\rho}}{\partial \underset{\sim}{\xi}^2}\right)\delta\underset{\sim}{\xi} - \\ &- g_n \underline{\boldsymbol{n}} \cdot \left(\delta\frac{\partial^2 \underline{\rho}}{\partial \underset{\sim}{\xi}^2}\Delta\underset{\sim}{\xi} + \Delta\frac{\partial^2 \underline{\rho}}{\partial \underset{\sim}{\xi}^2}\delta\underset{\sim}{\xi}\right) - g_n \Delta\underset{\sim}{\xi}^T\left(\underline{\boldsymbol{n}} \cdot \frac{\partial^3 \underline{\rho}}{\partial \underset{\sim}{\xi}^3}\right)\delta\underset{\sim}{\xi} + \\ &+ \left[g_n\left(\delta\frac{\partial \underline{\rho}}{\partial \underset{\sim}{\xi}} + \frac{\partial^2 \underline{\rho}}{\partial \underset{\sim}{\xi}^2}\delta\underset{\sim}{\xi}\right) \cdot \frac{\partial \underline{\rho}}{\partial \underset{\sim}{\xi}}^T \bar{\underset{\approx}{A}} - \delta g_n \underset{\approx}{I}\right]\left(\underline{\boldsymbol{n}} \cdot \Delta\frac{\partial \underline{\rho}}{\partial \underset{\sim}{\xi}} + \underset{\approx}{H}\Delta\underset{\sim}{\xi}\right) + \\ &+ \left[g_n\left(\Delta\frac{\partial \underline{\rho}}{\partial \underset{\sim}{\xi}} + \frac{\partial^2 \underline{\rho}}{\partial \underset{\sim}{\xi}^2}\Delta\underset{\sim}{\xi}\right) \cdot \frac{\partial \underline{\rho}}{\partial \underset{\sim}{\xi}}^T \bar{\underset{\approx}{A}} - \Delta g_n \underset{\approx}{I}\right]\left(\underline{\boldsymbol{n}} \cdot \delta\frac{\partial \underline{\rho}}{\partial \underset{\sim}{\xi}} + \underset{\approx}{H}\delta\underset{\sim}{\xi}\right)\Bigg\}\end{aligned}$$

Example

- Use penalty method to enforce Dirichlet BC
- Use penalty method to enforce contact constraints
- First, detect contact elements
- Second, construct updated residual vector and tangent matrix

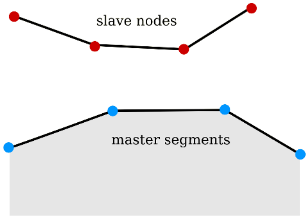


Contact between two elements

Detection

Introduction

- Important and time consuming part
- With which master segment the slave node can/will come in contact?
- Need to know it in advance
- To reduce time:
  - Bounding boxes for the global search
  - Maximal distance of detection



Slave and master

Introduction

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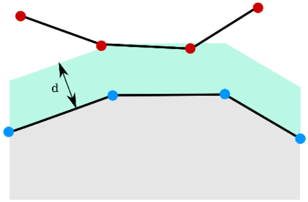

Slave in close zone

Introduction

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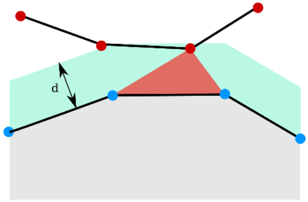

NTS contact element

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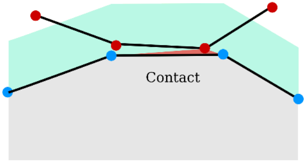

Contact occurs

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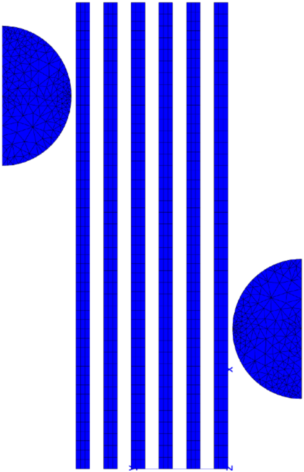

Multi-plate contact
Z-set/Zébulon

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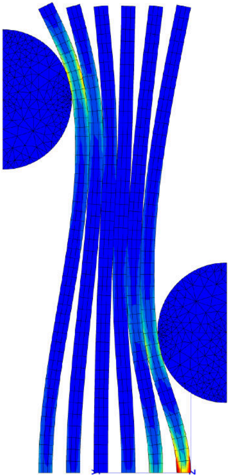

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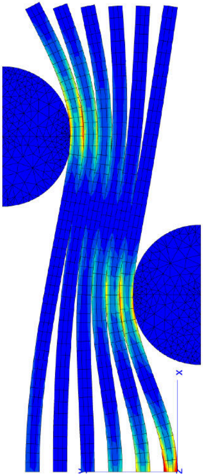

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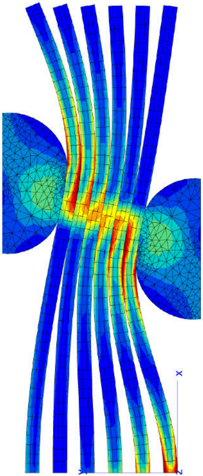

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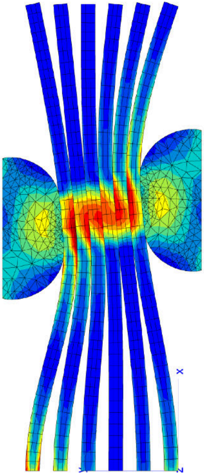

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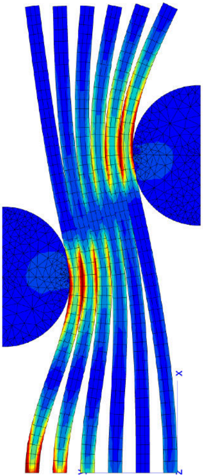

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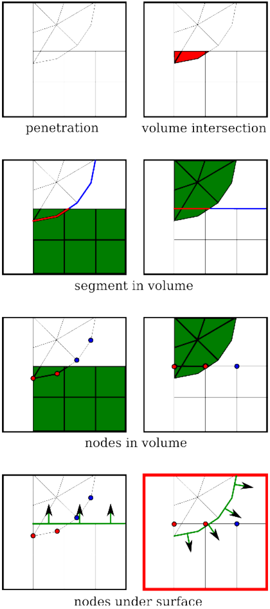
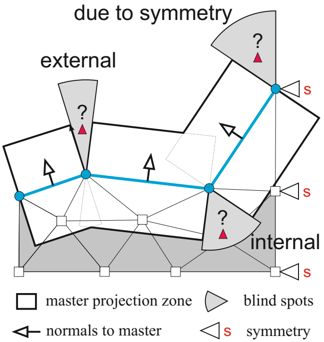
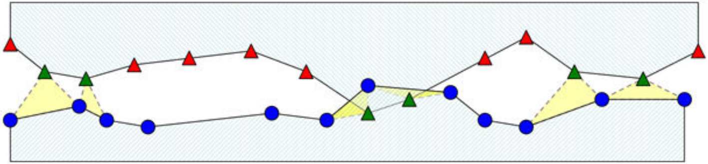


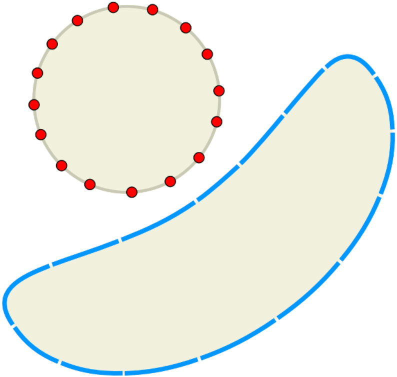
- Growth rate $O(N \times M)$
- Not robust
- Blind spots
- Slow





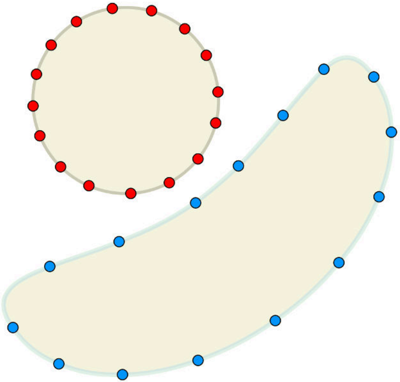
Closest node approach

- Node-to-segment detection ⇒ iterative solution



Closest node approach

- Node-to-segment detection ⇒ iterative solution
- Detection based on the closest node:
  1. find the closest master node;
  2. find a projection on segments adjacent to this node.
- Widely accepted simplification
- Simple treatment of blind spots



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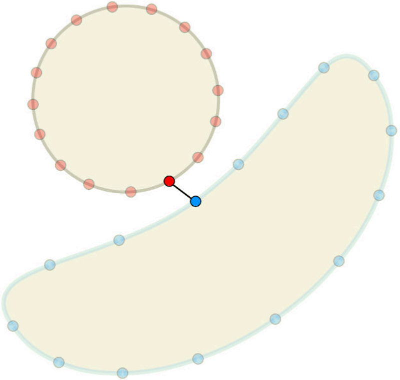

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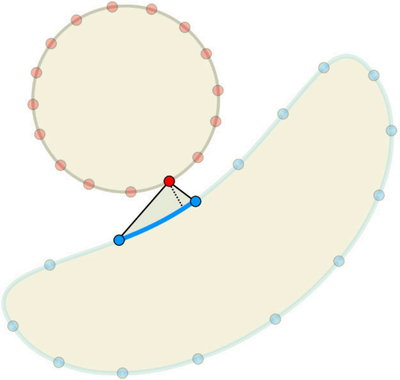
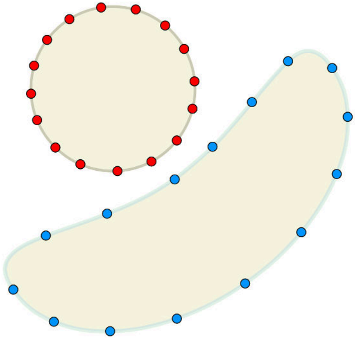


Acceleration of detection

- "Blind" algorithm ⇒ one-by-one



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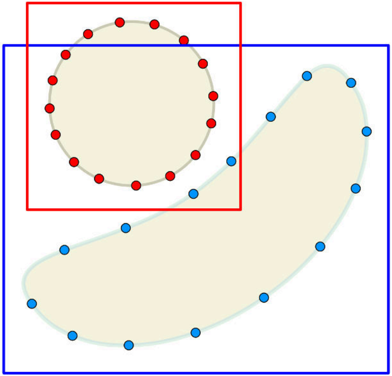
- "Blind" algorithm ⇒ one-by-one
- Reduce the number of elements to check:
  - Detection bounding box - intersection of master and slave bounding boxes



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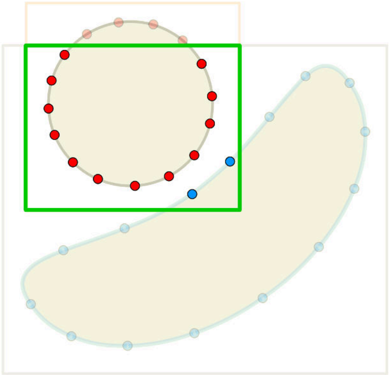

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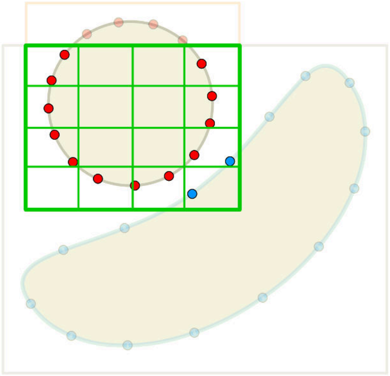
  - Distribute nodes in buckets (cells) ⇒ Bucket sort method [1]
  - For each slave node check only in several buckets



[1] Benson, Hallquist, 1991

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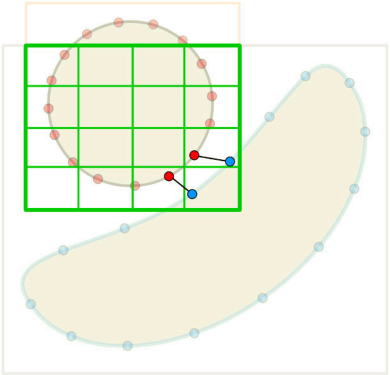

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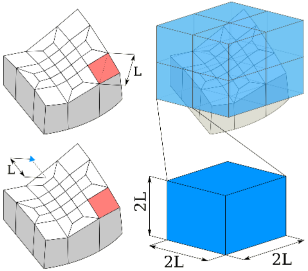
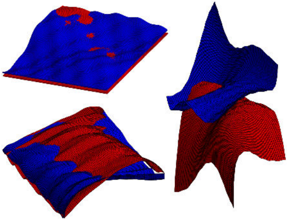
[1] Benson, Hallquist, 1991

Acceleration of detection

- Strong connection between:
  - finite element mesh L,
  - maximal detection distance L,
  - bucket's size $2L$.
- User friendly algorithm
- Complexity $O(N)$



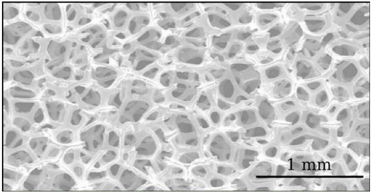
Relations between the master mesh, maximal detection distance and bucket's dimensions



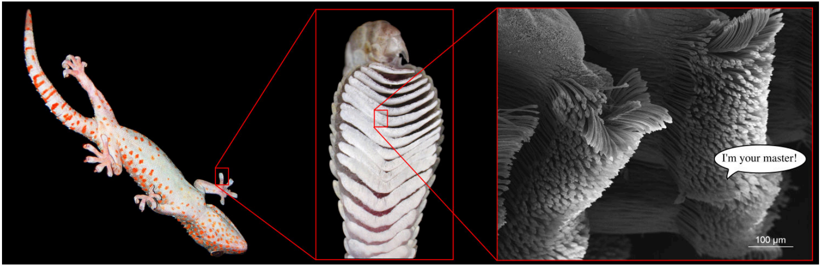
Meshes for numerical tests

Unknown master-slave

- Master-slave may be unknown in advance:
  - complex geometry;
  - large sliding;
  - self-contact.



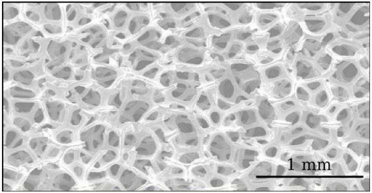
Nickel foam microstructure



Microstructure of gecko's adhesive toe
(adapted from Autumn Lab, Lewis& Clark Colledge, Portland, Oregon)

Unknown master-slave

- Master-slave may be unknown in advance:
  - complex geometry;
  - large sliding;
  - self-contact.



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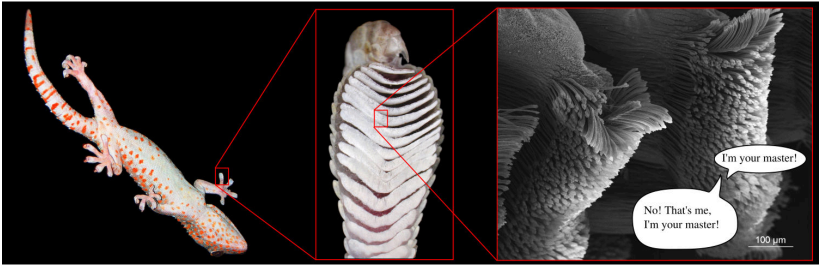

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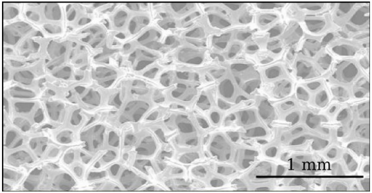

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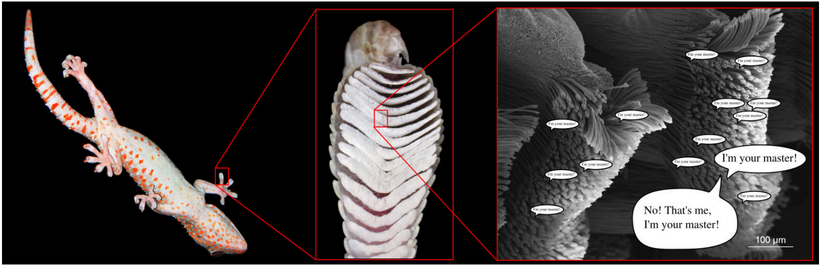
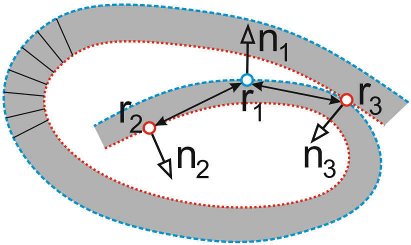
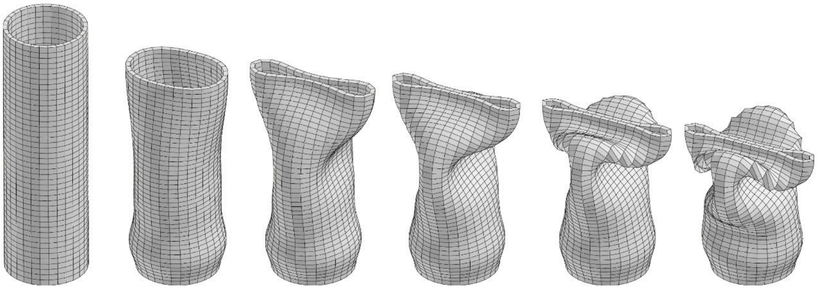


Microstructure of gecko's adhesive toe
(adapted from Autumn Lab, Lewis& Clark Colledge, Portland, Oregon)

Unknown master-slave

- Unknown master-slave
- The same algorithm
- Account of the nodal normals





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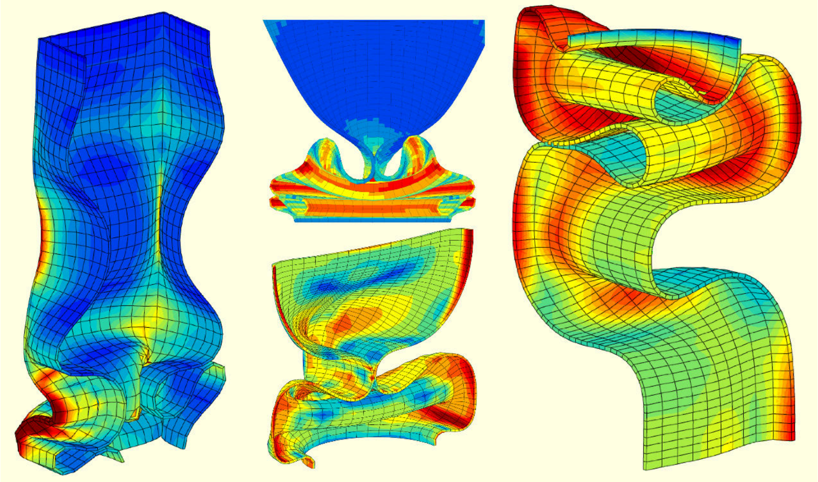
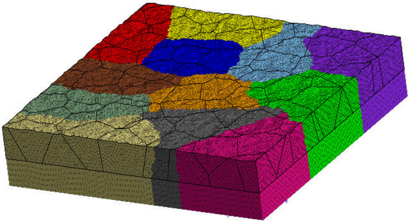
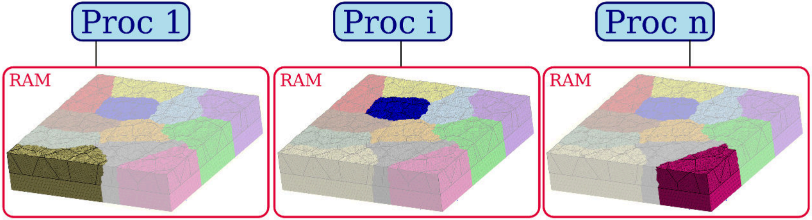


Finite Element Analyses of post-buckling behavior of thin-walled structures (self-contact, finite strain plasticity) Zset/Zébulon

Parallelization

- Distributed memory computer architecture
- ⇒ Distributed contact surface
- No information about the entire contact surface





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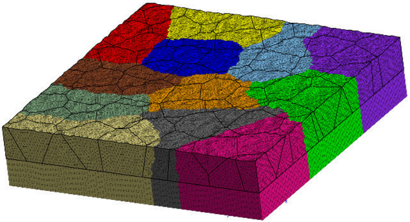
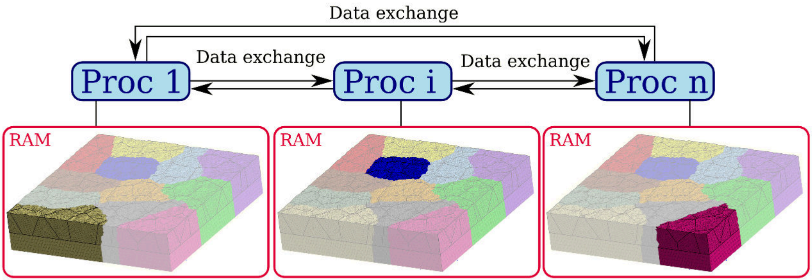
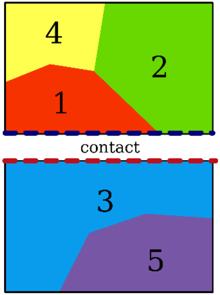
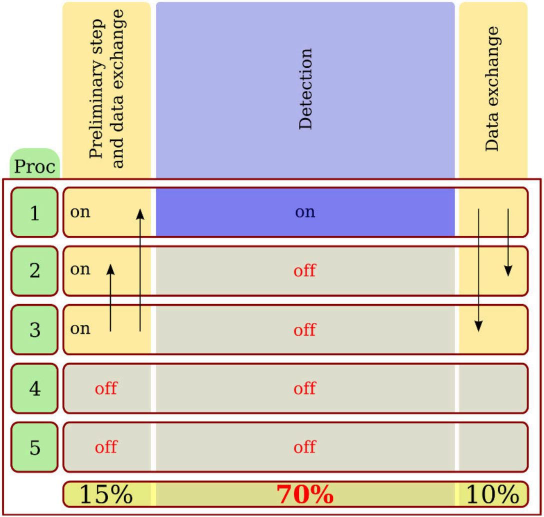




Detection by a single CPU

- SDMR
  Single Detection, Multiple Resolution
- Not optimal
- Simple data exchange





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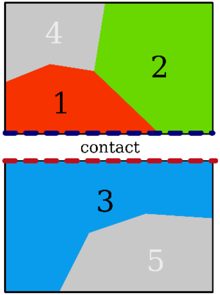
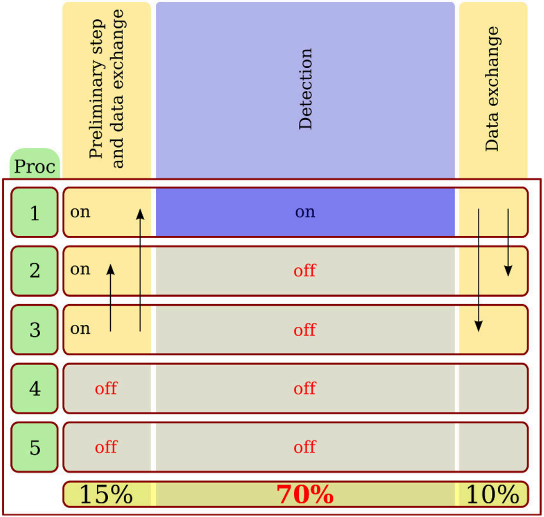
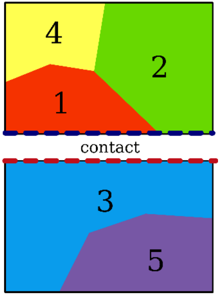
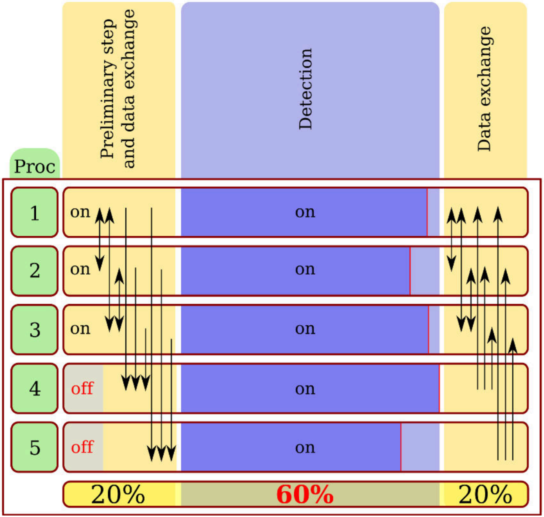




Distributed detection

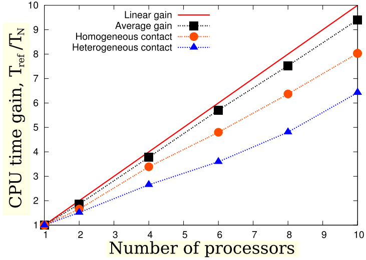
- MDMR
  Multiple Detection,
  Multiple Resolution
- More optimal
- Complex data exchange

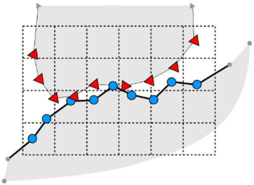




Distributed detection II

- Global detection bounding box
- Split into N equal overlapping parts
- One bucket overlap
- Test





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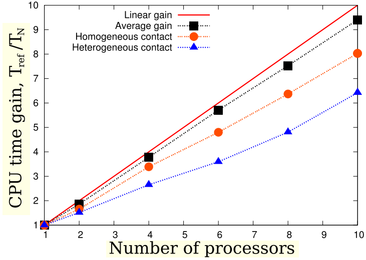

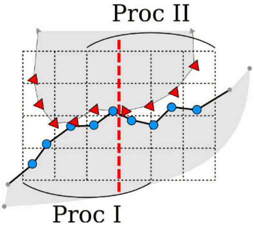

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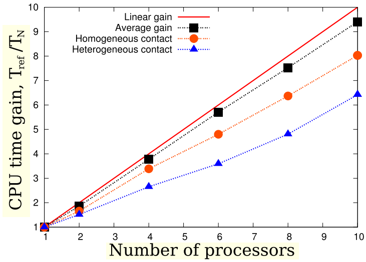

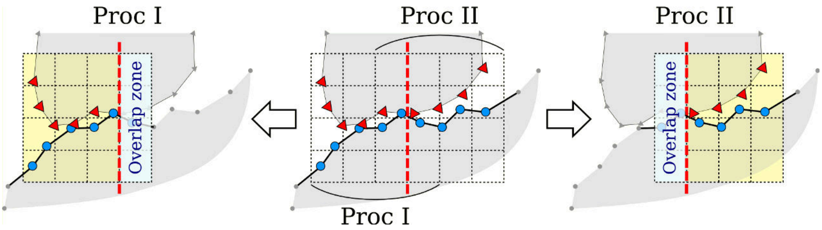

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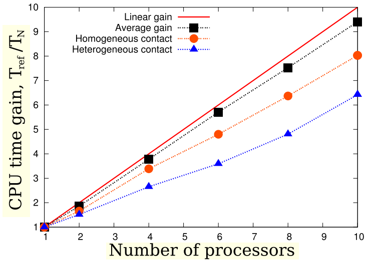
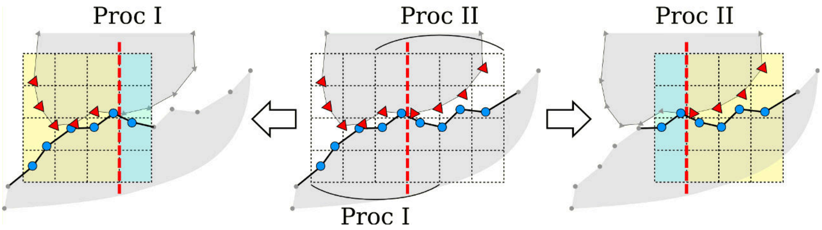
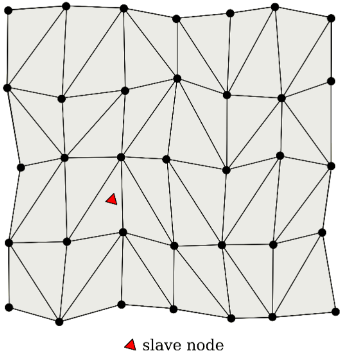




When simple detection fails...

Methods based on the closest node detection are not robust

- Idea:
  - find the closest master node
  - find a projection on the adjacent segments
- Counterexample
- Closest segment is not always attached to the closest node
  - not regular mesh
  - triangular mesh
- Carefull use or improvement



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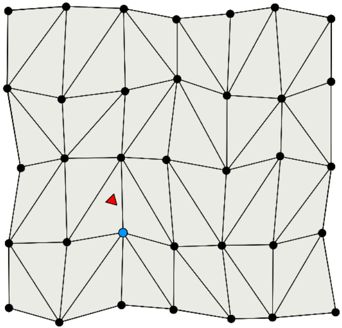

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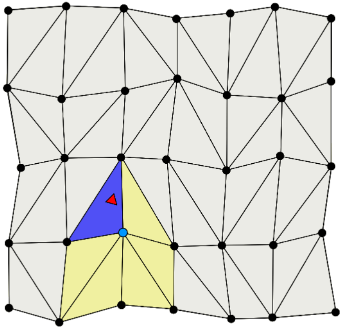

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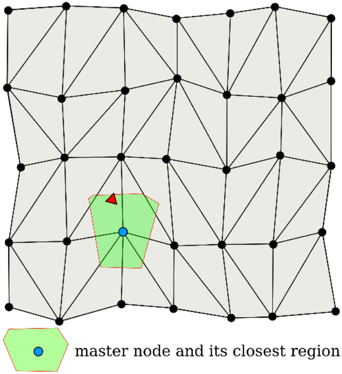

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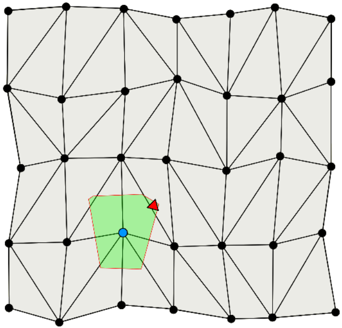

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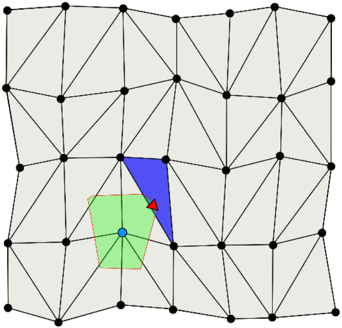

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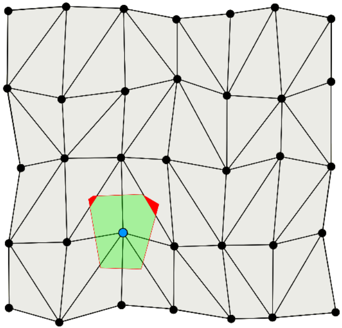

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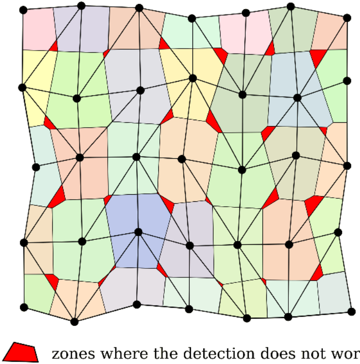

zones where the detection does not work

Methods based on the closest node detection are not robust

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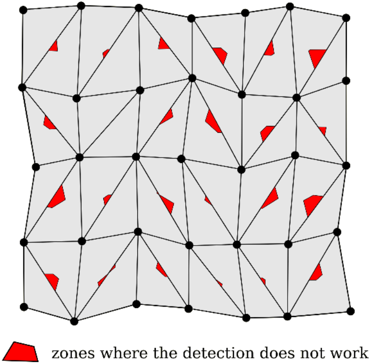

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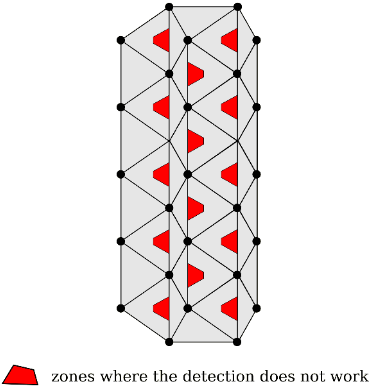
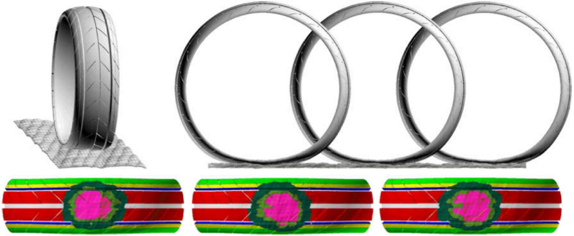
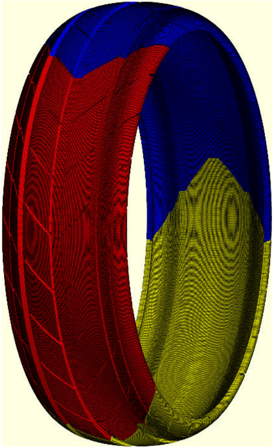


Example I

- Tyre-road problem
  - Tyre – 100 000 slave nodes
  - Road – 200 000 master segments
  - Detection **1.5-2** seconds



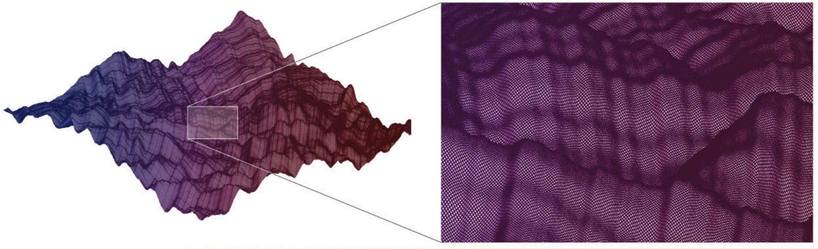
Contact elements for different loads
Zset/Zébulon



FE mesh of a tire
550 000 nodes, 105 000 slave nodes
Zset/Mesher

Example II

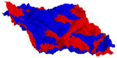
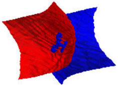
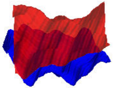
- Two curved surfaces in contact
  - 10^6 against 10^6 contact nodes
  - All-to-all $T_{\text{all-to-all}}$ **>180 hours**
  - Bucket sort performance depends on geometry:



FE mesh of one of the contacting surfaces
Zset/Mesher

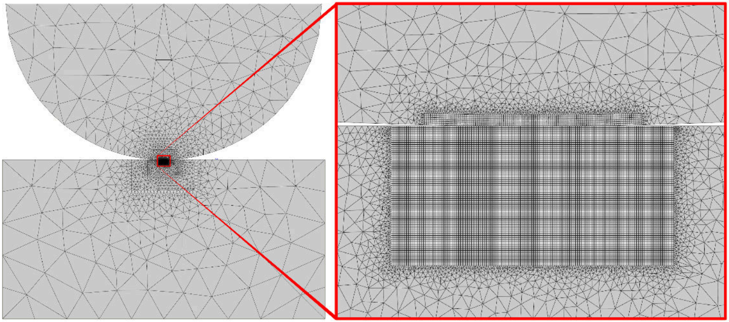
Example II

- Two curved surfaces in contact
  - 10^6 against 10^6 contact nodes
  - All-to-all $T_{\text{all-to-all}}$ **>180 hours**
  - Bucket sort performance depends on geometry:

Geometry	Nodes in bounding box	CPU time	Gain, $T_{\text{all-to-all}}/T_{\text{bucket}}$
	2 100 000	35 minutes	>300 times
	340 000	1 minute	>10 500 times
	50 000	4 seconds	>160 000 times

Particularities: mesh and convergence

- Strong **mesh refinement** is required
  - especially at **unknown edges** of contact zones



Typical mesh for fretting analysis [L. Sun, H. Proudhon, G. Cailletaud, 2011]

2D ~ 30 000 DoFs,   3D ~ 5 000 000 DoFs

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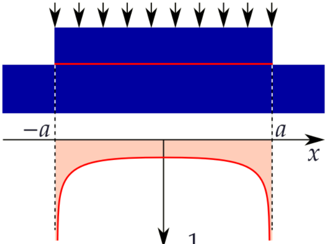
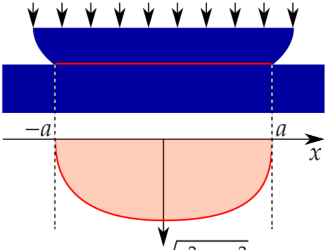


$$\sigma_n \sim \frac{1}{\sqrt{a^2 - x^2}}$$

$$\sigma_n \xrightarrow[x\to a]{} -\infty \qquad \left|\frac{\partial \sigma_n}{\partial x}\right| \xrightarrow[x\to a]{} \infty$$



$$\sigma_n \sim \sqrt{a^2 - x^2}$$

$$\left|\frac{\partial \sigma_n}{\partial x}\right| \xrightarrow[x\to a]{} \infty$$

Infinite contact pressure and/or its derivative

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- Strong **mesh refinement** is required
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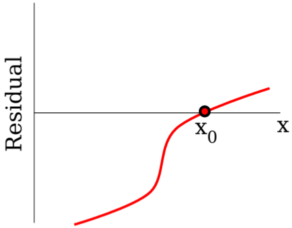
- **Slow change** of boundary conditions:
  - strong non-linearities of contact / friction problems
  - non-uniqueness of solution for frictional problems

Infinite looping



Initial guess $R(x_0, f_0) = 0$

Particularities: mesh and convergence

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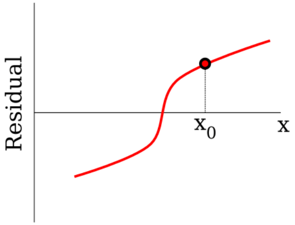

Too rapid change in boundary conditions $R(x_0, f_1) \neq 0$

Particularities: mesh and convergence

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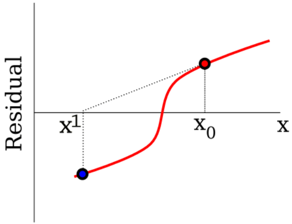

Iterations of Newton-Raphson method

$$R(x_0, f_1) + \frac{\partial R}{\partial x}\Big|_{x_0} \delta x = 0 \rightarrow \delta x = -\frac{\partial R}{\partial x}\Big|_{x_0}^{-1} R(x_0, f_1) \rightarrow x^1 = x_0 + \delta x$$

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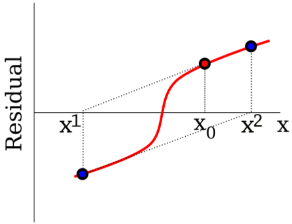

Iterations of Newton-Raphson method

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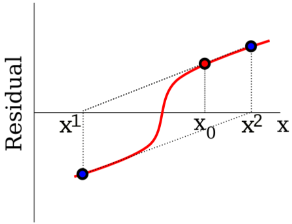

Infinite looping

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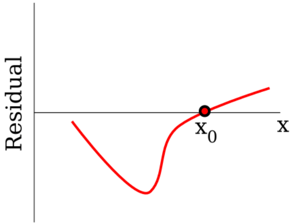
Convergence to a "false" solution



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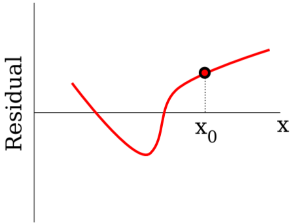
Convergence to a "false" solution



Too rapid change in boundary conditions $R(x_0, f_1) \neq 0$

Particularities: mesh and convergence

- Strong **mesh refinement** is required
  - especially at **unknown edges** of contact zones
- **Slow change** of boundary conditions:
  - strong non-linearities of contact / friction problems
  - non-uniqueness of solution for frictional problems

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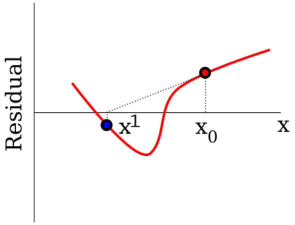

Iterations of Newton-Raphson method

$$R(x_0, f_1) + \left.\frac{\partial R}{\partial x}\right|_{x_0} \delta x = 0 \rightarrow \delta x = -\left.\frac{\partial R}{\partial x}\right|_{x_0}^{-1} R(x_0, f_1) \rightarrow x^1 = x_0 + \delta x$$

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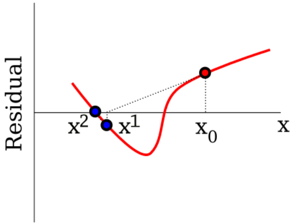

Iterations of Newton-Raphson method

$$R(x^1, f_1) + \frac{\partial R}{\partial x}\Big|_{x^1} \delta x = 0 \rightarrow \delta x = - \frac{\partial R}{\partial x}\Big|_{x^1}^{-1} R(x^1, f_1) \rightarrow x^2 = x^1 + \delta x$$

Particularities: mesh and convergence

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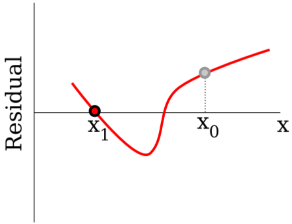
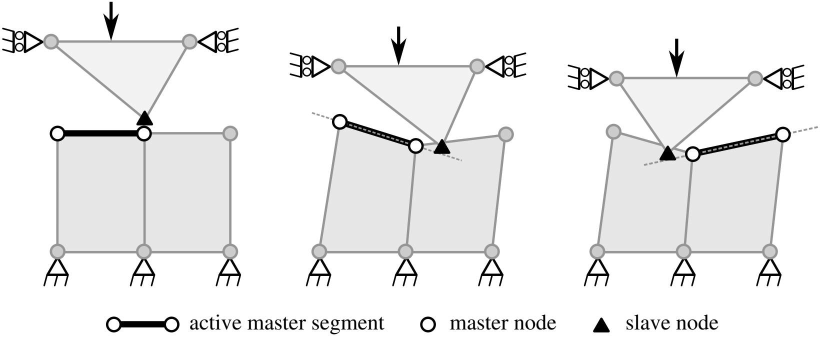


Convergence, but is it a "true" solution ?

Convergence problems: examples

- **Infinite looping, e.g.**

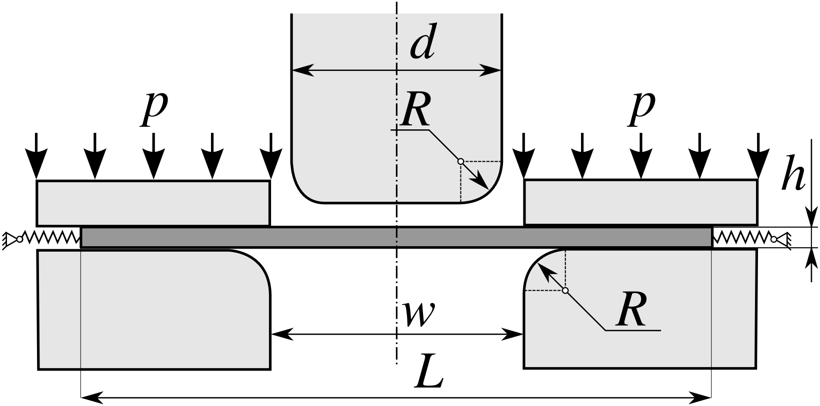


- Change of the contact state (contact/non-contact, stick/slip)
- Interplay between stiffness, friction and augmented Lagrangian coefficients[1]
- Combination of non-linearities (e.g., plasticity+contact)

Alart P., Journal de Mathématiques Pures et Appliqués 76 (1997)

Convergence problems: examples

- Simulation of a deep drawing problem
- Finite strain plasticity + frictional contact



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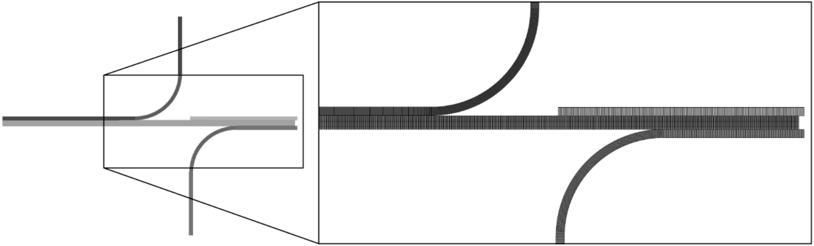

Convergence problems: examples

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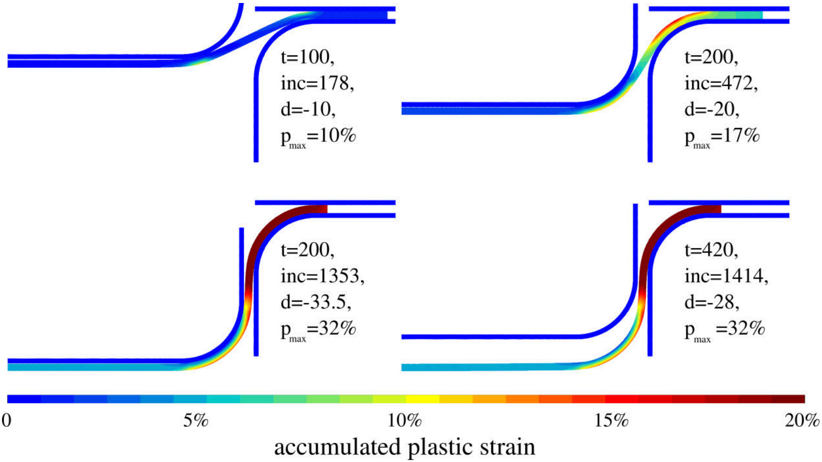

Convergence problems: examples

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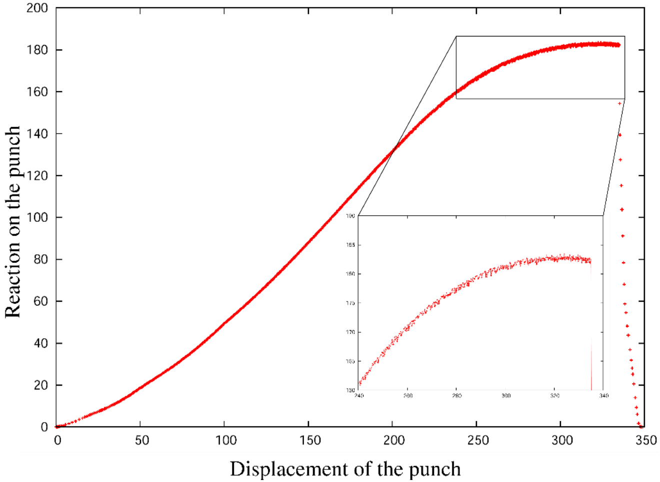
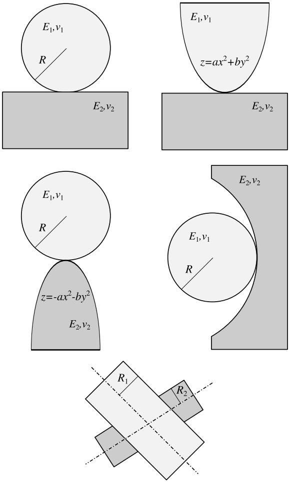


Cylinder-plane frictional contact

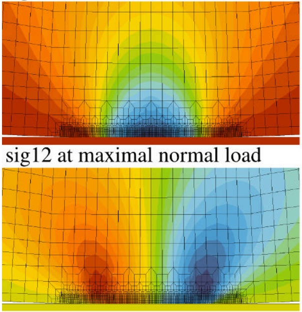
- Non-conservative problem, history of loading is crucial

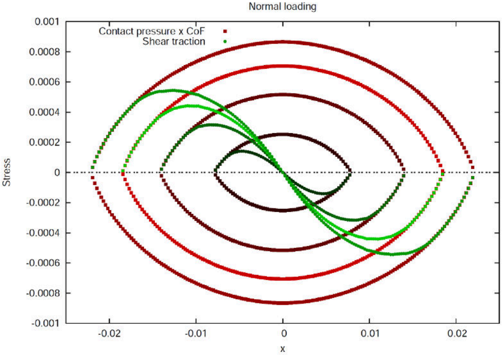


- Non-conservative problem, history of loading is crucial

sig22 at maximal normal load

sig12 at maximal normal load

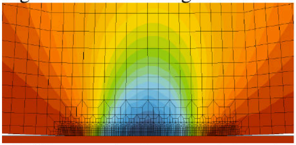




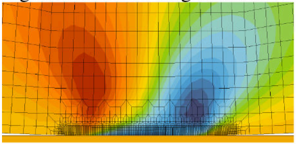
Press in 100 increments, $u_z \sim t^2$

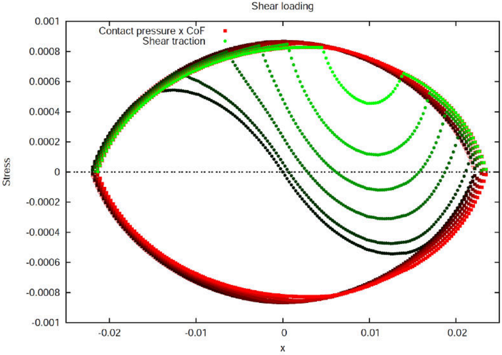
- Non-conservative problem, history of loading is crucial

sig22 at maximal tangential load



sig12 at maximal tangential load

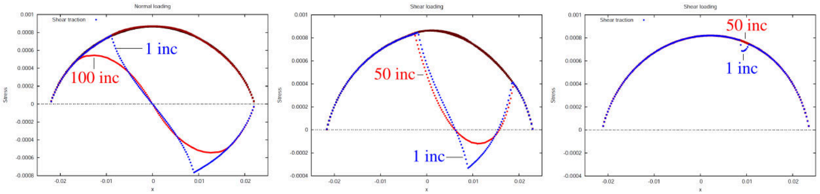




Shift in 100 increments, $u_z \sim t$

Cylinder-plane frictional contact

- Non-conservative problem, history of loading is crucial

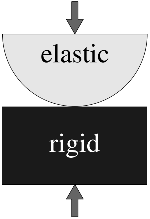


Comparison with: press in 1 increment, shift in 2 increments

Before sticking, every point of the contact interface has to pass through the slip zone. It is impossible when loaded too fast.

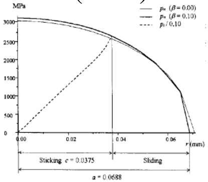
Warning friction!

- For dissimilar materials, the *friction matters* even in normal contact
- The problem is thus path-dependent, the B.C. should be changed slowly

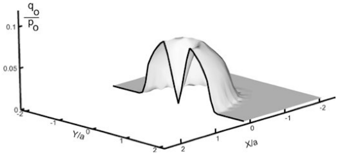


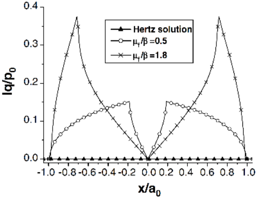
Erroneous solutions

(1999)

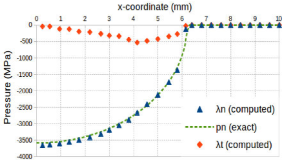


(2003)



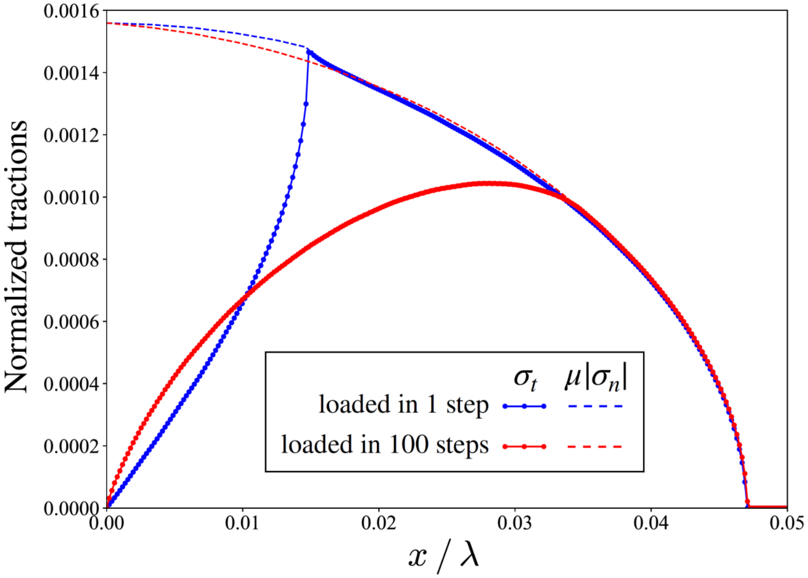


(2008)



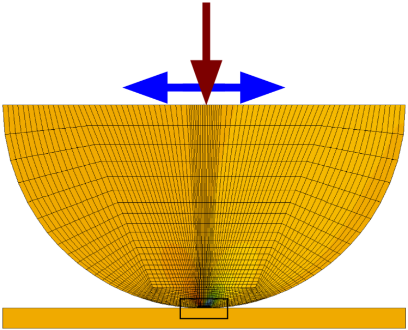
(2017)

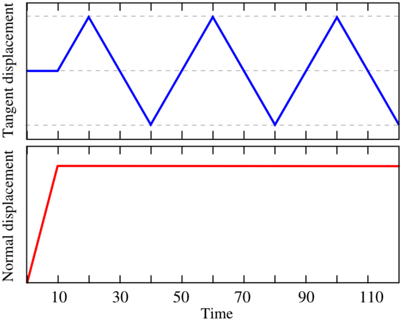
Warning friction!



[1] A.G. Shvarts, PhD thesis, MINES ParisTech (2019)

Sphere-plane frictional contact: cycling





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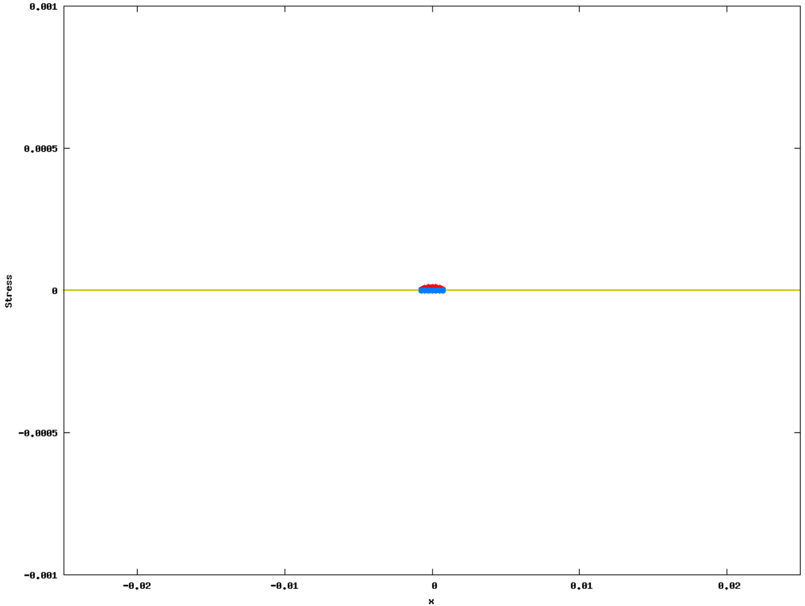

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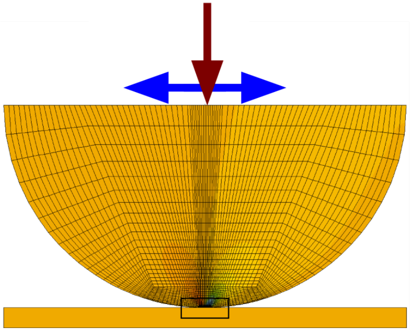

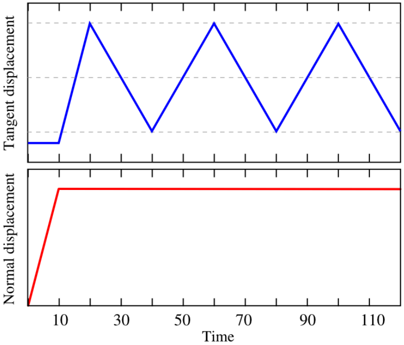

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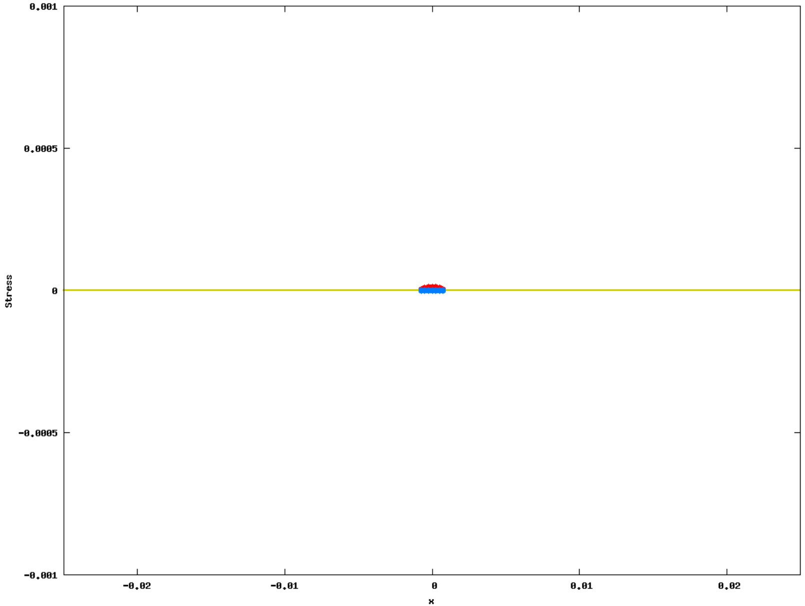
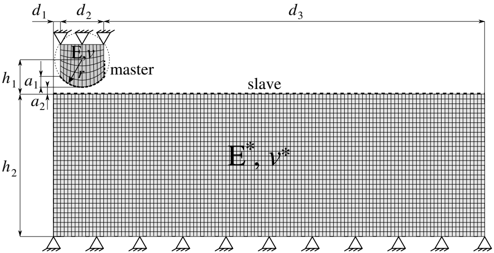


Shallow ironing test

- Deformable-on-deformable frictional sliding
- Results obtained by different groups[1,2,3,4,5,6] differ significantly
- Local and global friction coefficients may differ



[1] Fischer K. A., Wriggers P., "Mortar based frictional contact formulation for higher order interpolations using the moving friction cone", Computer Methods in Applied Mechanics and Engineering, vol. 195, p. 5020-5036, 2006.

[2] Hartmann S., Oliver J., Cante J. C., Weyler R., Hernández J. A., "A contact domain method for large deformation frictional contact problems. Part 2: Numerical aspects", Computer Methods in Applied Mechanics and Engineering, vol. 198, p. 2607-2631, 2009.

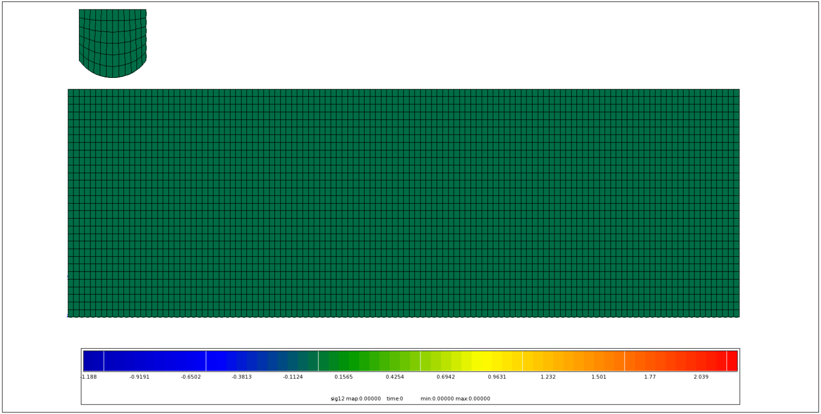
[3] Yastrebov V. A., "Computational contact mechanics: geometry, detection and numerical techniques", Thèse CdM & Onera, 2011.

[4] Kudawoo A. D., "Problèmes industriels de grande dimension en mécanique numérique du contact : performance, fiabilité et robustesse", Thèse @ LMA & LAMSID, 2012.

[5] Poulios K., Renard Y., "A non-symmetric integral approximation of large sliding frictional contact problems of deformable bodies based on ray-tracing", soumis, 2014.

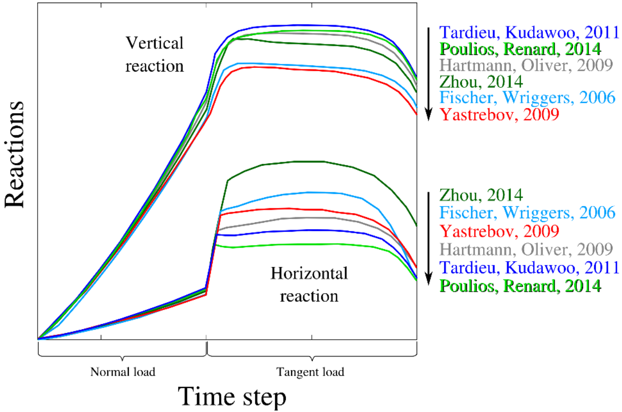
[6] Zhou Lei's blog, http://kt2008plus.blogspot.de

Shallow ironing test



Shallow ironing test

- No agreement between authors
- Dif. authors used dif. meshes (quadrilateral lin./sq., triangular lin.)
- Dif. authors used either finite or infinitesimal strain formulation



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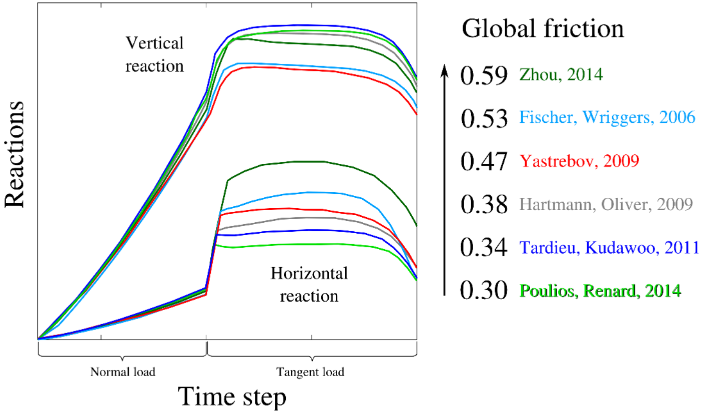


Local coefficient of friction $\mu_l = 0.3$

Examples of contact problems

With analytical solution

- ★ linear elasticity
- ★ with/without friction

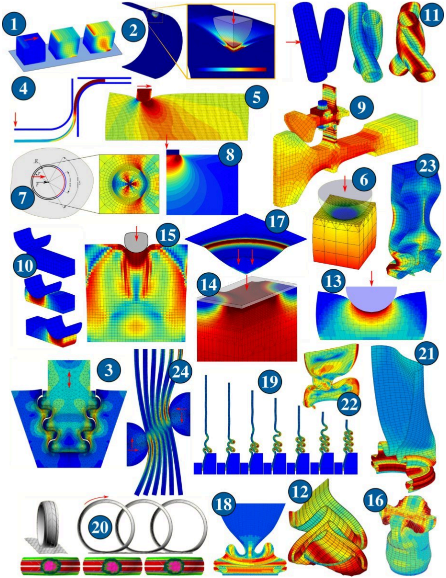
From literature

- ★ post-buckling 2D
- ★ finite strains
- ★ elasticity / plasticity
- ★ with/without friction

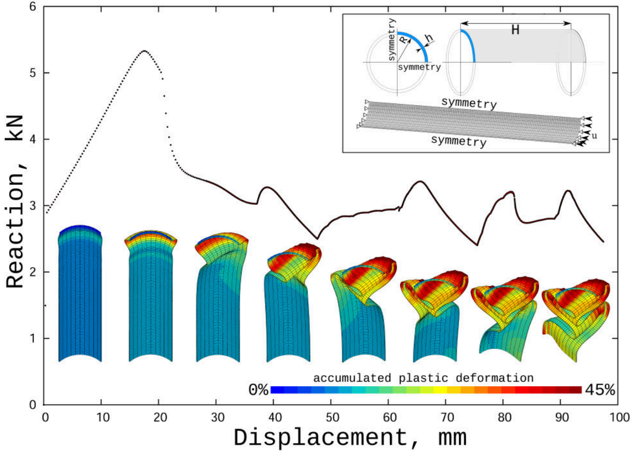
New

- ★ multi-contacts
- ★ post-buckling 3D
- ★ finite strains
- ★ elasticity / plasticity
- ★ with/without friction

Z-set | Material & structure analysis suite



Self-contact problem



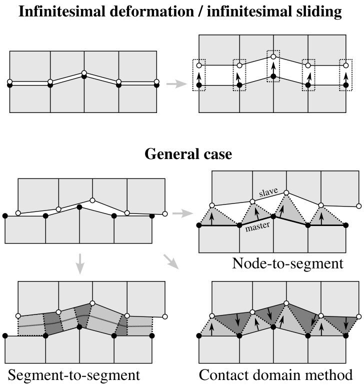
Finite element analysis of a post-buckling behavior of a thin walled tube

Collection of non-linearities: buckling instability, self-contact, finite strain plasticity

Z-set Material & structure analysis suite

Reading

- It's just a tip of the "Computational Contact Mechanics" iceberg
- Contact discretization and integration
- Smoothing techniques
- Energy conservative methods for dynamics



Contact discretization techniques

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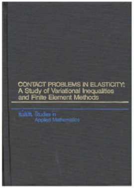
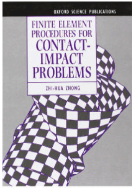
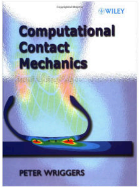


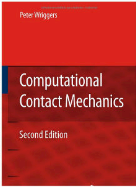
Kikuchi, Oden (1988)



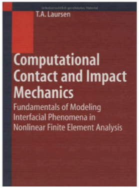
Zhong (1993)



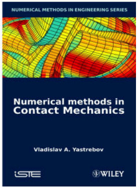
Wriggers (2002)



Wriggers, 2^{nd}ed. (2006)



Laursen (2002)



Yastrebov (2013)

$\mathcal{L}_a(x, \lambda)$ Thank you for your attention!

www.yastrebov.fr