

Computational Contact Mechanics

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Centre des Matériaux, Evry/Paris, France*



Nonlinear Computational Mechanics @ Mines Paris
November 19, 2025



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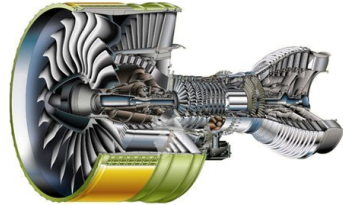
Outline

- Introduction
- Basics of Contact and Friction
- Towards a weak form
- Optimization methods
- Resolution algorithm
- Examples

Introduction

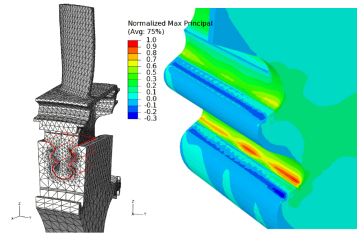
Industrial and natural contact problems

1 Assembled parts, e.g. engines



Aircraft's engine GP 7200

www.safran-group.com



[1] M. W. R. Savage

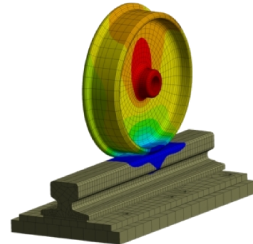
J. Eng. Gas Turb. Power, 134:012501 (2012)

Industrial and natural contact problems

- 1 Assembled parts, e.g. engines
- 2 Railroad contacts



High speed train TGV www.sncf.com



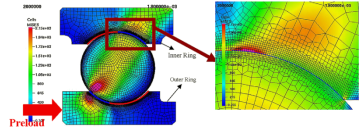
Wilde/ANSYS wildeanalysis.co.uk

Industrial and natural contact problems

- 1 Assembled parts, e.g. engines
- 2 Railroad contacts
- 3 Gears and bearings



Bearings
www.skf.com



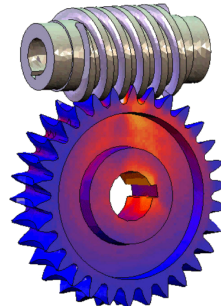
[1] F. Massi, J. Rocchi, A. Culla, Y. Berthier
Mech. Syst. Signal Pr., 24:1068-1080 (2010)

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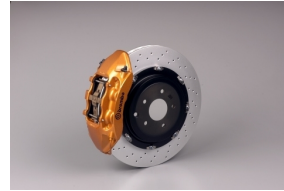
Helical gear www.tpg.com.tw



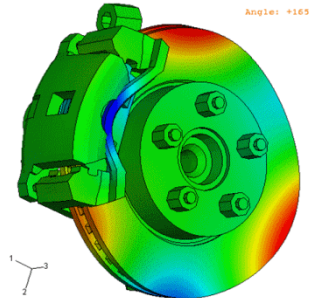
www.mscsoftware.com

Industrial and natural contact problems

- 1 Assembled parts, e.g. engines
- 2 Railroad contacts
- 3 Gears and bearings
- 4 Breaking systems



Assembled breaking system
www.brembo.com



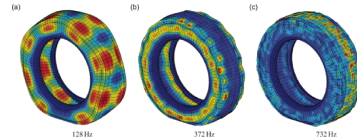
www.mechanicalengineeringblog.com

Industrial and natural contact problems

- 1 Assembled parts, e.g. engines
- 2 Railroad contacts
- 3 Gears and bearings
- 4 Breaking systems
- 5 Tire-road contact



Tire-road contact www.michelin.com



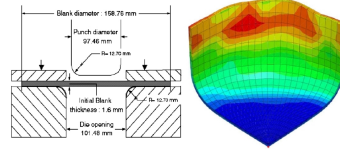
[1] M. Brinkmeier, U. Nackenhorst, S. Petersen, O. von Estorff, *J. Sound Vib.*, 309:20-39 (2008)

Industrial and natural contact problems

- 1 Assembled parts, e.g. engines
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- 4 Breaking systems
- 5 Tire-road contact
- 6 Metal forming



Deep drawing www.thomasnet.com



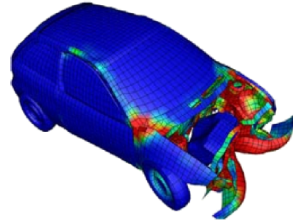
[1] G. Rousselier, F. Barlat, J. W. Yoon
Int. J. Plasticity, 25:2383-2409 (2009)

Industrial and natural contact problems

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- 7 Crash tests



Crash-test www.porsche.com



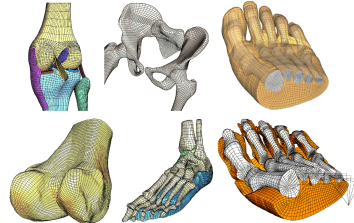
[1] O. Klyavin, A. Michailov, A. Borovkov www.fea.ru

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- 6 Metal forming
- 7 Crash tests
- 8 Biomechanics



Human articulations
www.sportssupplements.net



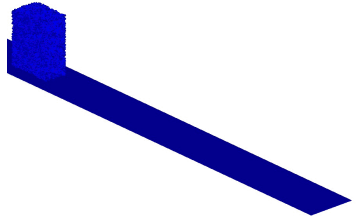
J. A. Weiss, University of Utah
Musculoskeletal Research Laboratories

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- 8 Biomechanics
- 9 Granular materials



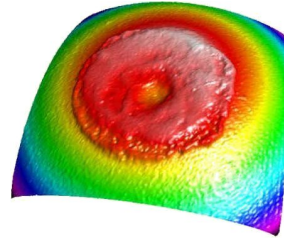
Sand dunes www.en.wikipedia.org



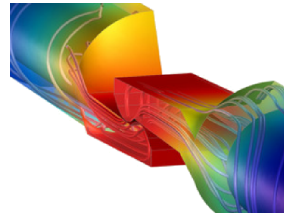
E. Azema et al, LMGC90

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- 10 Electric contacts



Damage at electric contact zone
www.taicaan.com



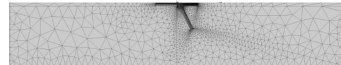
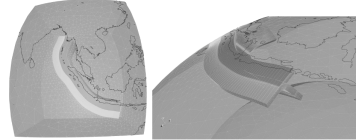
Simulation of electric current
www.comsol.com

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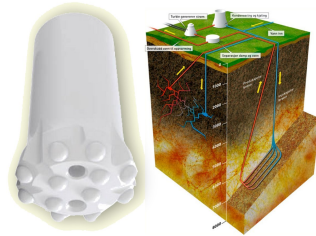
San-Andreas fault, by M. Rightmire
www.sciencedude.ocregister.com



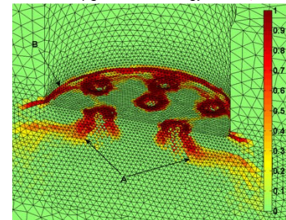
[1] J.D. Garaud, L. Fleitout, G. Cailletaud
Colloque CSMA (2009)

Industrial and natural contact problems

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- 9 Granular materials
- 10 Electric contacts
- 11 Tectonic motions
- 12 Deep drilling



Drill Bit tool *Robit Rocktools*;
extraction of geothermal energy (*SINTEF, NTNU*)



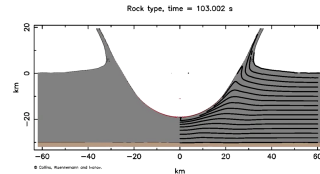
[1] T. Saksala, *Int. J. Numer. Anal. Meth. Geomech.* (2012)

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- 12 Deep drilling
- 13 Impact and fragmentation



Impact crater, Arizona
www.MrEclipse.com et maps.google.com



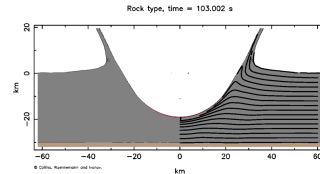
Simulation of formation of Copernicus crater
Yue Z., Johnson B. C., et al. Projectile remnants in central peaks of lunar impact craters. *Nature Geo* 6 (2013)

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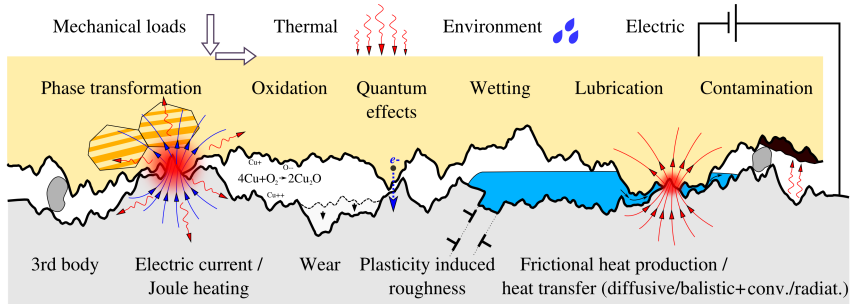
Impact crater, Arizona
www.MrEclipse.com et maps.google.com



Simulation of formation of Copernicus crater
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Physical and mathematical complexity

- Contact interface is hard to observe in situ
- Plenty of phenomena happen in the interface
- Strong thermo-mechanical or fluid-solid coupling in sliding
- Mathematical formulation is also non-trivial, hard to handle analytically
- **Robust and accurate computational framework is needed**



Interface between two solids in contact

Basics of Contact and Friction

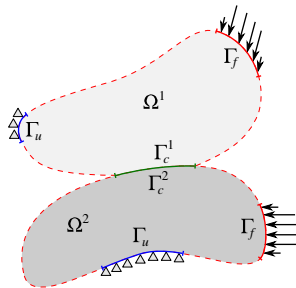
Equilibrium and contact conditions

■ Balance of momentum

$$\begin{cases} \nabla \cdot \underline{\underline{\sigma}} + \underline{\underline{f}}_V = 0 & \text{in } \Omega_{1,2} \\ \underline{\underline{\sigma}} \cdot \underline{\underline{n}} = \underline{\underline{t}}_0 & \text{on } \Gamma_f \\ \underline{\underline{u}} = \underline{\underline{u}}_0 & \text{on } \Gamma_u \\ \text{?} & \text{on } \Gamma_c \end{cases}$$

■ Frictionless contact conditions (*intuitive*)

- 1 No penetration
- 2 No adhesion
- 3 No shear transfer



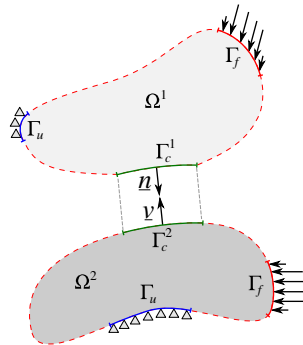
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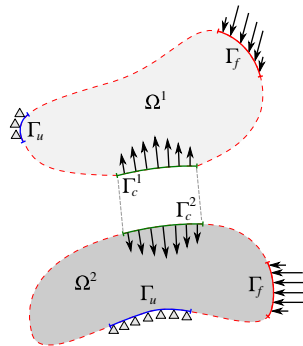
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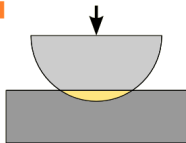
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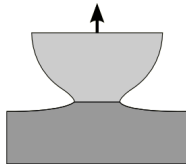
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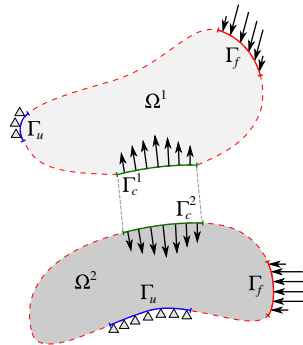
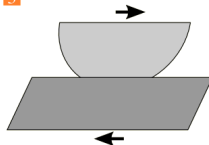
1



2



3



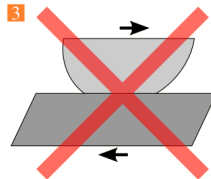
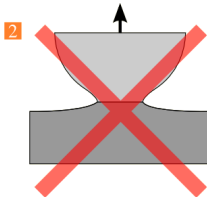
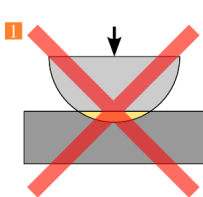
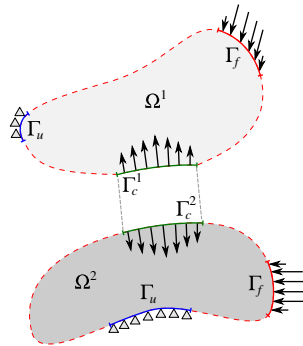
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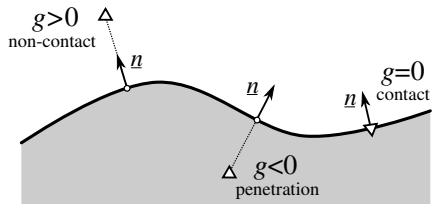
Gap function

■ Gap function g

- gap = - penetration
- asymmetric function
- defined for
 - separation $g > 0$
 - contact $g = 0$
 - penetration $g < 0$
- governs normal contact

■ Master and slave split

Gap function is determined for all slave points with respect to the master surface



Gap between a slave point and a master surface

Gap function

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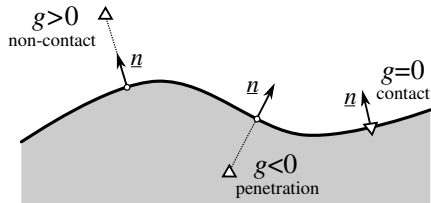
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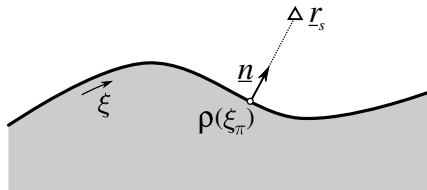
■ Normal gap

$$g_n = \underline{n} \cdot [\underline{r}_s - \underline{\rho}(\xi_\pi)]$$

$\underline{n}$ is a unit normal vector, $\underline{r}_s$ slave point, $\underline{\rho}(\xi_\pi)$ projection point on the master surface



Gap between a slave point and a master surface



Definition of the normal gap

Consider existence and uniqueness



Frictionless or normal contact conditions

■ No penetration

Always non-negative gap

$$g \geq 0$$

■ No adhesion

Always non-positive contact pressure

$$\sigma_n^* \leq 0$$

■ Complementarity condition

Either zero gap and non-zero pressure, or non-zero gap and zero pressure

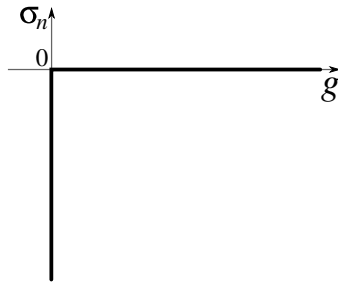
$$g \sigma_n = 0$$

■ No shear transfer (automatically)

$$\underline{\sigma}_t^{**} = 0$$

$$\sigma_n^* = (\underline{\underline{\sigma}} \cdot \underline{n}) \cdot \underline{n} = \underline{\underline{\sigma}} : (\underline{n} \otimes \underline{n})$$

$$\underline{\sigma}_t^{**} = \underline{\underline{\sigma}} \cdot \underline{n} - \sigma_n \underline{n} = \underline{n} \cdot \underline{\underline{\sigma}} \cdot (\underline{I} - \underline{n} \otimes \underline{n})$$



Scheme explaining normal contact conditions

Frictionless or normal contact conditions

■ No penetration

Always non-negative gap

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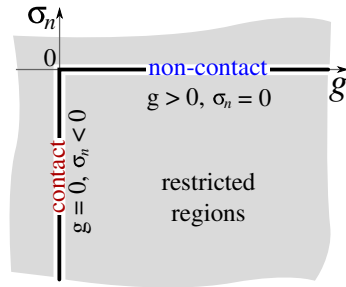
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Improved scheme explaining normal contact conditions

Frictionless or normal contact conditions

In **mechanics**:

Normal contact conditions

$\equiv$

Frictionless contact conditions

$\equiv$

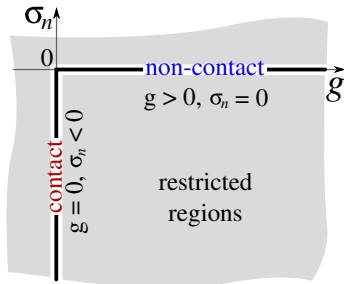
Hertz^[1]_{DE}-Signorini^[2]_{IT} conditions

$\equiv$

Hertz^[1]_{DE}-Signorini^[2]_{IT}-Moreau^[3]_{FR} conditions

also known in **optimization theory** as

Karush^[4]_{US}-Kuhn^[5]_{US}-Tucker^[6]_{CA} (KKT) conditions



Improved scheme explaining normal contact conditions

$$g \geq 0, \quad \sigma_n \leq 0, \quad g \sigma_n = 0$$

¹Heinrich Rudolf Hertz (1857–1894) a German physicist who first formulated and solved the frictionless contact problem between elastic ellipsoidal bodies.

²Antonio Signorini (1888–1963) an Italian mathematical physicist who gave a general and rigorous mathematical formulation of contact constraints.

³Jean Jacques Moreau (1923–2014) a French mathematician who formulated a non-convex optimization problem based on these conditions and introduced pseudo-potentials in contact mechanics.

⁴William Karush (1917–1997), ⁵Harold William Kuhn (1925–2014) American mathematicians,

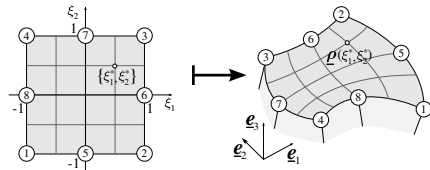
⁶Albert William Tucker (1905–1995) a Canadian mathematician.

Relative sliding

Recall:

- Convective coordinate in parent space
 $\xi_i \in (-1; 1)$
- Mapping to real space

$$\underline{\rho}(\xi_1, \xi_2, t) = \sum_{i=1}^8 N^i(\xi_1, \xi_2) \underline{\rho}^i(t)$$



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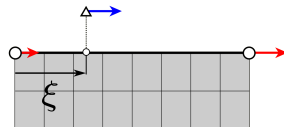
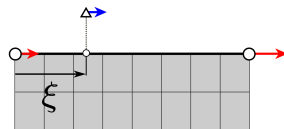
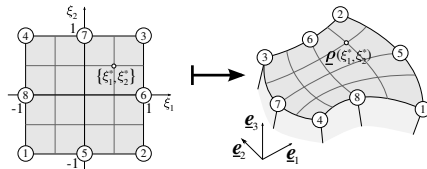
■ Tangential slip velocity $\underline{v}_t$

must take into account:

- only tangential component
- relative rigid body motion
- master's deformation

$$\underline{v}_t = \frac{\partial \underline{\rho}}{\partial \xi_1} \dot{\xi}_1 + \frac{\partial \underline{\rho}}{\partial \xi_2} \dot{\xi}_2$$

where $\partial \underline{\rho} / \partial \xi_i$ are the tangent vectors of the local basis and $\dot{\xi}_i$ are the convective coordinates.



Relative slip between a slave point and a deformable master surface

Relative sliding

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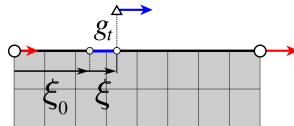
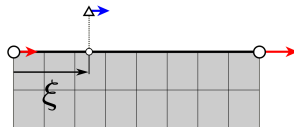
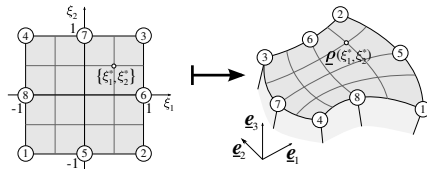
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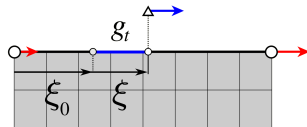
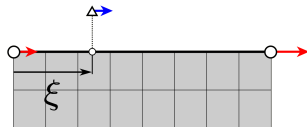
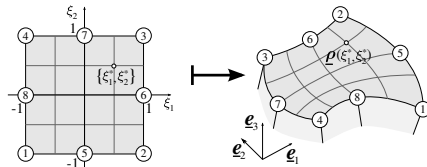
■ Tangential slip velocity $\underline{v}_t$

must take into account:

- only tangential component
- relative rigid body motion
- master's deformation

$$\underline{v}_t = \frac{\partial \underline{\rho}}{\partial \xi_1} \dot{\xi}_1 + \frac{\partial \underline{\rho}}{\partial \xi_2} \dot{\xi}_2$$

where $\partial \underline{\rho} / \partial \xi_i$ are the tangent vectors of the local basis and $\dot{\xi}_i$ are the convective coordinates.



Relative slip between a slave point and a deformable master surface

Relative sliding

Recall:

- Convective coordinate in parent space
 $\xi_i \in (-1; 1)$
- Mapping to real space

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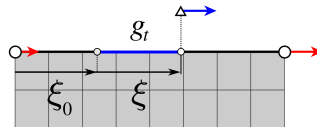
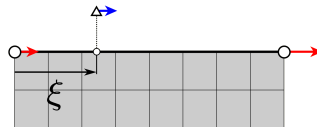
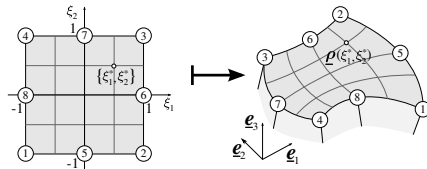
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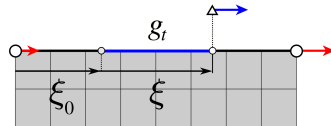
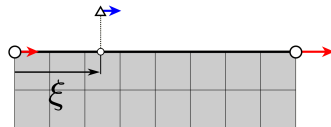
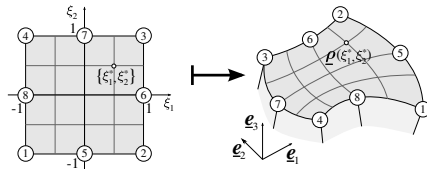
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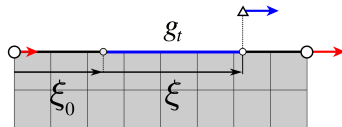
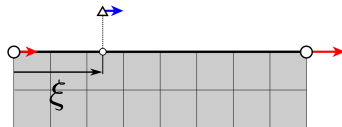
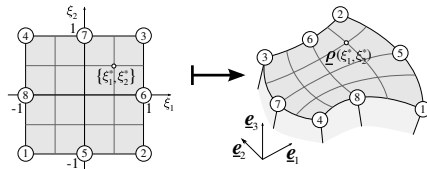
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Relative slip between a slave point and a deformable master surface

Relative sliding: example

Consider a one-dimensional example:

P is a projection of A on segment BC .

$$x_P = \xi x_C + (1 - \xi)x_B \quad (1)$$

Velocity of the projection point

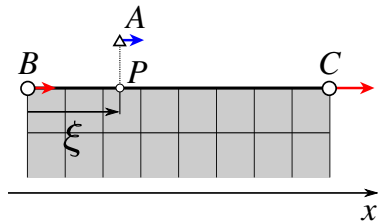
$$\dot{x}_P = \underbrace{\xi \dot{x}_C + (1 - \xi) \dot{x}_B}_{\frac{\partial x_P}{\partial t}} + \underbrace{(x_C - x_B) \dot{\xi}}_{\frac{\partial x_P}{\partial \xi} \dot{\xi}}$$

Subtract the velocity of point x_P for fixed ξ

$$v_t = \dot{x}_P - \frac{\partial x_P}{\partial t} = (x_C - x_B) \dot{\xi} = \frac{\partial x}{\partial \xi} \dot{\xi}$$

Compute tangential slip increment

$$\Delta g_t^{n+1} = \left. \frac{\partial x}{\partial \xi} \right|_{\xi^n} (\xi^{n+1} - \xi^n)$$



Example of a one-dimensional relative slip

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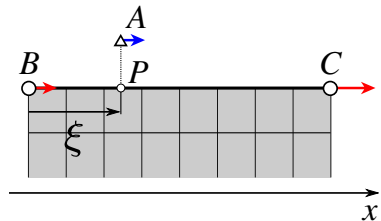
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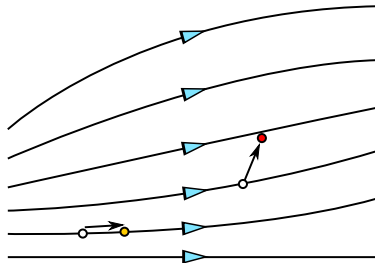
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Example of a one-dimensional relative slip



Fisherman's analogy: observing sea flow around the boat.
Lie derivative: the change of a vector field along the change of another vector field

Amontons-Coulomb's friction

■ **No contact** $g > 0, \sigma_n = 0$

■ **Stick** $|\underline{v}_t| = 0$

Inside slip surface / Coulomb's cone

$$f = |\underline{\sigma}_t| - \mu |\sigma_n| < 0$$

■ **Slip** $|\underline{v}_t| > 0$

On slip surface / Coulomb's cone

$$f = |\underline{\sigma}_t| - \mu |\sigma_n| = 0$$

■ **Complementary condition**

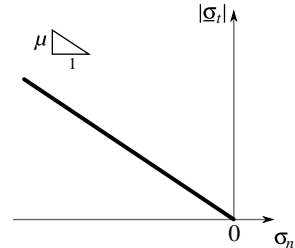
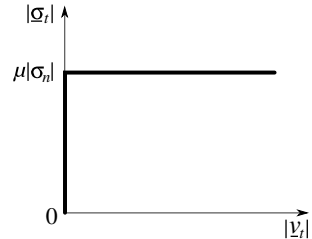
One is zero another one is not or vice versa

$$|\underline{v}_t| (|\underline{\sigma}_t| - \mu |\sigma_n|) = 0$$

■ **Direction of friction**

Shear and sliding are collinear

$$\underline{v}_t \parallel \underline{\sigma}_t$$



Scheme explaining frictional contact conditions

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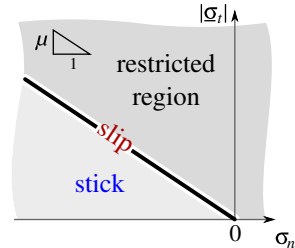
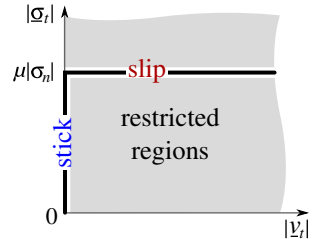
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$$\underline{v}_t \parallel \underline{\sigma}_t$$



Improved scheme explaining frictional contact conditions

Amontons-Coulomb's friction

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- **Stick** $|\underline{v}_t| = 0$

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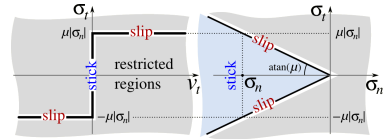
One is zero another one is not or vice versa

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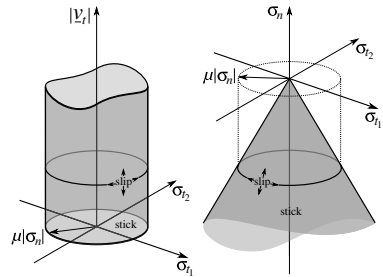
- **Direction of friction**

Shear and sliding are collinear

$$\underline{v}_t \parallel \underline{\sigma}_t$$



Scheme of 2D frictional contact

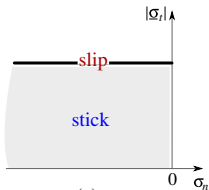


Scheme of 3D frictional contact

$$|\underline{v}_t| \geq 0, \quad |\underline{\sigma}_t| - \mu |\sigma_n| \leq 0, \quad |\underline{v}_t| (|\underline{\sigma}_t| - \mu |\sigma_n|) = 0, \quad \frac{\underline{\sigma}_t}{|\underline{\sigma}_t|} = - \frac{\underline{v}_t}{|\underline{v}_t|}$$

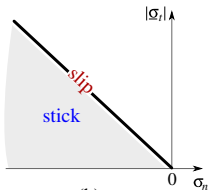
More friction laws

• Static criteria



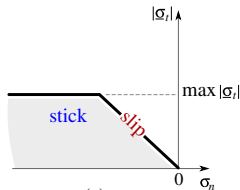
(a)

(a) Tresca



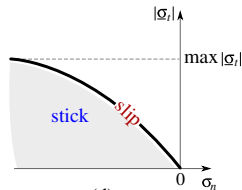
(b)

(b) Amontons-Coulomb



(c)

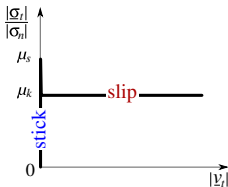
(c) Coulomb-Orowan



(d)

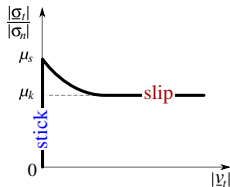
(d) Shaw

• Kinetic criteria



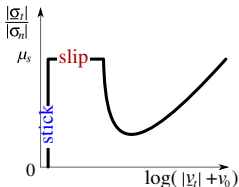
(a)

(a,b) velocity weakening

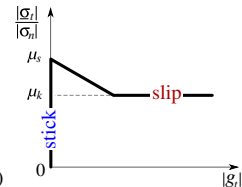


(b)

(c) velocity weakening-strengthening



(c)



(d)

(d) linear slip weakening

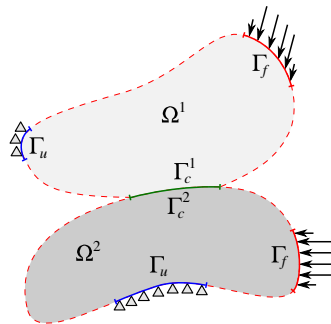
• μ_s static and μ_k kinetic coefficients of friction.

Towards a weak form

From strong to a weak form

- Balance of momentum and boundary conditions

$$\nabla \cdot \underline{\underline{\sigma}} + \underline{\underline{f}}_v = 0 \quad \text{in } \Omega = \Omega_1 \cup \Omega_2 \quad + \text{B.C.}$$



Two solids in contact

From strong to a weak form

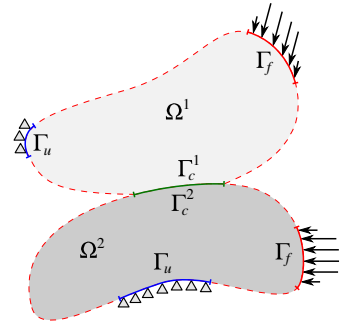
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- Balance of virtual works



$$\boxed{\int_{\partial\Omega} \underline{n} \cdot \underline{\underline{\sigma}} \cdot \delta \underline{u} \, d\Gamma} + \int_{\Omega} \left[\underline{\underline{f}}_v \cdot \delta \underline{u} - \underline{\underline{\sigma}} \cdot \delta \nabla \underline{u} \right] d\Omega = 0$$



Two solids in contact

From strong to a weak form

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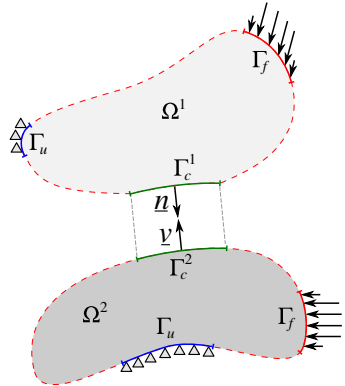
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$$\int_{\partial\Omega} \underline{\underline{n}} \cdot \underline{\underline{\sigma}} \cdot \delta \underline{\underline{u}} d\Gamma =$$

$$\int_{\bar{\Gamma}_c^1} \underline{\underline{n}} \cdot \underline{\underline{\sigma}} \cdot \delta \underline{\underline{\rho}} d\bar{\Gamma}_c^1 + \int_{\bar{\Gamma}_c^2} \underline{\underline{\nu}} \cdot \underline{\underline{\sigma}} \cdot \delta \underline{\underline{r}} d\bar{\Gamma}_c^2 + \int_{\Gamma_f} \underline{\underline{\sigma}}_0 \cdot \delta \underline{\underline{u}} d\Gamma_f$$



Two solids in contact

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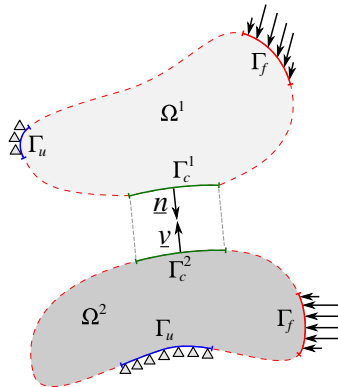
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$$\int_{\partial\Omega} \underline{\underline{n}} \cdot \underline{\underline{\sigma}} \cdot \delta \underline{\underline{u}} d\Gamma =$$

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$$= \int_{\bar{\Gamma}_c^1} \underline{\underline{n}} \cdot \underline{\underline{\sigma}} \cdot \delta (\underline{\underline{\rho}} - \underline{\underline{r}}) d\bar{\Gamma}_c^1 = \int_{\bar{\Gamma}_c^1} \left(\sigma_n \delta g_n + \underline{\underline{\sigma}}_t^T \delta \underline{\underline{\xi}} \right) d\bar{\Gamma}_c^1$$



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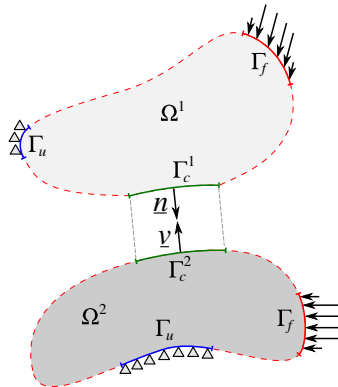


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$$\int_{\Omega} \underline{\underline{\sigma}} : \delta \nabla \underline{\underline{u}} d\Omega + \underbrace{\int_{\bar{\Gamma}_c^1} \left(\sigma_n \delta g_n + \underline{\underline{\sigma}}_t^T \delta \underline{\underline{\xi}} \right) d\bar{\Gamma}_c^1}_{\text{Contact term}} = \int_{\Gamma_f} \underline{\underline{\sigma}}_0 \cdot \delta \underline{\underline{u}} d\Gamma + \int_{\Omega} \underline{\underline{f}}_v \cdot \delta \underline{\underline{u}} d\Omega$$



Two solids in contact

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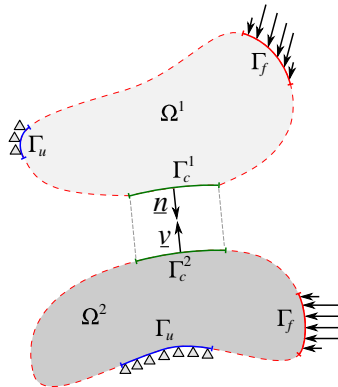


$$\underbrace{\int_{\Omega} \underline{\underline{\sigma}} \cdot \delta \nabla \underline{\underline{u}} \, d\Omega}_{\text{Change of the internal energy}} + \underbrace{\int_{\bar{\Gamma}_c^1} \left(\sigma_n \delta g_n + \underline{\underline{\sigma}}_t^T \delta \underline{\underline{\xi}} \right) d\bar{\Gamma}_c^1}_{\text{Contact term}} =$$

$$\underbrace{\int_{\Gamma_f} \underline{\underline{\sigma}}_0 \cdot \delta \underline{\underline{u}} \, d\Gamma}_{\text{Virtual work of external forces}} + \underbrace{\int_{\Omega} \underline{\underline{f}}_v \cdot \delta \underline{\underline{u}} \, d\Omega}_{\text{Virtual work of volume forces}}$$

- Functional **space**

$\delta \underline{\underline{u}}, \underline{\underline{u}} \in \mathbb{H}^1(\Omega)$ Sobolev space of the first order (function and its first derivative is square integrable) and $\underline{\underline{u}}$ satisfy boundary conditions



Two solids in contact

From strong to a weak form

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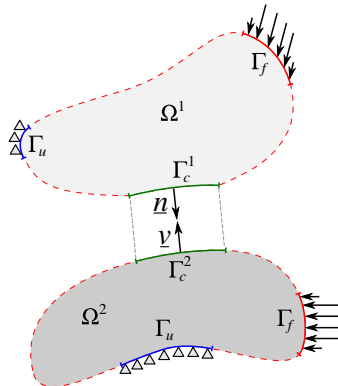


$$\underbrace{\int_{\Omega} \underline{\underline{\sigma}} \cdot \cdot \delta \nabla \underline{\underline{u}} d\Omega}_{\text{Change of the internal energy}} + \underbrace{\int_{\bar{\Gamma}_c^1} \left(\sigma_n \delta g_n + \underline{\underline{\sigma}}_t^T \delta \underline{\underline{\xi}} \right) d\bar{\Gamma}_c^1}_{\text{Contact term}} \geq$$

$$\underbrace{\int_{\Gamma_f} \underline{\underline{\sigma}}_0 \cdot \delta \underline{\underline{u}} d\Gamma}_{\text{Virtual work of external forces}} + \underbrace{\int_{\Omega} \underline{\underline{f}}_v \cdot \delta \underline{\underline{u}} d\Omega}_{\text{Virtual work of volume forces}}$$

- Functional **subspace**

$\delta \underline{\underline{u}}, \underline{\underline{u}} \in \mathbb{H}^1(\Omega)$ Sobolev space of the first order (function and its first derivative is square integrable) and $\underline{\underline{u}}$ satisfy boundary conditions and **contact conditions**, so we do optimization on a subset of $\mathbb{H}^1(\Omega)$.



Two solids in contact

Variational inequality

- Optimization problem for $F : \mathbb{V} \rightarrow \mathbb{R}$
- Find $u \in \mathbb{V}$ s.t. $\forall v \in \mathbb{V}: F(u) \leq F(v)$
- If $F \in C^1$ is convex then such minimizer u is a stationary point $F'|_u = 0$

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- In the limit

$$\lim_{\theta \rightarrow 0} \frac{F(u + \theta(v - u)) - F(u)}{\theta} \geq 0$$

Variational inequality

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- Find $u \in \mathbb{V}$ s.t. $\forall v \in \mathbb{V}: F(u) \leq F(v)$
- If $F \in C^1$ is convex then such minimizer u is a stationary point $F'|_u = 0$
- However, finding minimizer of F on a subset $\mathbb{K} \subset \mathbb{V}$ changes the story
- If $\mathbb{K}$ is convex, then if $u \in \mathbb{K}$ is a minimizer, $\forall v \in \mathbb{K}, \theta \in [0, 1] : F(u) \leq F(u + \theta(v - u))$
- In the limit

$$\lim_{\theta \rightarrow 0} \frac{F(u + \theta(v - u)) - F(u)}{\theta} = F'(u)(v - u) \geq 0$$

Variational inequality

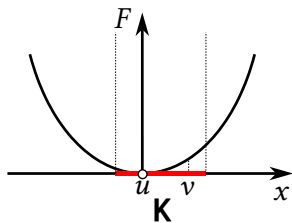
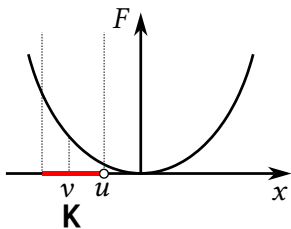
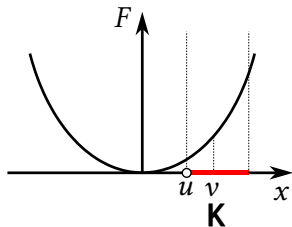
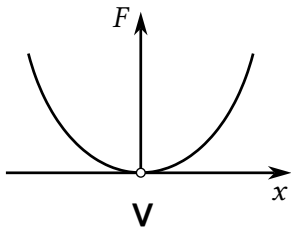
- Optimization problem for $F : \mathbb{V} \rightarrow \mathbb{R}$
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- Variational inequality for minimizer $u \in \mathbb{K} \subset \mathbb{V}$:

$$F'(u)(v - u) \geq 0, \quad \forall v \in \mathbb{K}$$

Example of variational inequality



Minimize $F(x)$ for $x \in K \subset \mathbb{R}$, then the minimizer u satisfies

$$F'(u)(v - u) \geq 0, \quad \forall v \in K$$

Variational inequality and a simplification

- Constrained minimization problem (variational inequality)^[1,2]

$$\int_{\Omega} \underline{\underline{\sigma}} \cdot \delta \nabla \underline{\underline{u}} d\Omega + \int_{\bar{\Gamma}_c^1} \underline{\underline{\sigma}}_t^T \delta \underline{\underline{\xi}} d\bar{\Gamma}_c^1 \geq \int_{\Gamma_f} \underline{\underline{\sigma}}_0 \cdot \delta \underline{\underline{u}} d\Gamma + \int_{\Omega} \underline{\underline{f}}_v \cdot \delta \underline{\underline{u}} d\Omega, \quad \underline{\underline{u}} \in \mathbb{L}, \delta \underline{\underline{u}} \in \mathbb{K}$$

$$\mathbb{L} = \left\{ \underline{\underline{u}} \in \mathbb{H}^1(\Omega) \mid \underline{\underline{u}} = \underline{\underline{u}}_0 \text{ on } \Gamma_u, g_n(\underline{\underline{u}}) \geq 0 \text{ on } \Gamma_c \right\}$$

$$\mathbb{K} = \left\{ \delta \underline{\underline{u}} \in \mathbb{H}^1(\Omega) \mid \delta \underline{\underline{u}} = 0 \text{ on } \Gamma_u, g_n(\delta \underline{\underline{u}}) \geq 0 \text{ on } \Gamma_c \right\}$$

[1] Duvaut, G. and Lions, J.L., 1972. *Les inéquations en mécanique et en physique*. Dunod, Paris, 1972

[2] Duvaut, G. and Lions, J.L., 1976. *Inequalities in mechanics and physics*, Springer

Variational inequality and a simplification

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- Use optimization theory to convert to

$$\int_{\Omega} \underline{\sigma} \cdot \delta \nabla \underline{u} \, d\Omega + \int_{\Gamma_c^1} \underbrace{C(\sigma_n, \sigma_t, g_n, \underline{\xi}, \delta \underline{u})}_{\text{Contact term}^*} \, d\Gamma_c^1 = \int_{\Gamma_f} \underline{\sigma}_0 \cdot \delta \underline{u} \, d\Gamma + \int_{\Omega} \underline{f}_v \cdot \delta \underline{u} \, d\Omega,$$

Unconstrained functional sub-spaces

$$\mathbb{L} = \left\{ \underline{u} \in \mathbb{H}^1(\Omega) \mid \underline{u} = \underline{u}_0 \text{ on } \Gamma_u \right\}$$

$$\mathbb{K} = \left\{ \delta \underline{u} \in \mathbb{H}^1(\Omega) \mid \delta \underline{u} = 0 \text{ on } \Gamma_u \right\}$$

Contact term* is defined on the *potential contact zone* Γ_c^1 .

[1] Duvaut, G. and Lions, J.L., 1972. *Les inéquations en mécanique et en physique*. Dunod, Paris, 1972

[2] Duvaut, G. and Lions, J.L., 1976. *Inequalities in mechanics and physics*, Springer

Optimization methods

Optimization methods

Functional to be minimized $F(\mathbf{x})$ under constraint $g(\mathbf{x}) \geq 0$

- Penalty method
- Lagrange multipliers method
- Augmented Lagrangian method

Optimization methods

Functional to be minimized $F(\mathbf{x})$ under constraint $g(\mathbf{x}) \geq 0$

■ Penalty method

- New functional

$$F_p(\mathbf{x}) = F(\mathbf{x}) + \epsilon \langle -g(\mathbf{x}) \rangle^2 = F(\mathbf{x}) + \begin{cases} 0, & \text{if } g(\mathbf{x}) \geq 0 \quad \text{non-contact} \\ \epsilon g^2(\mathbf{x}), & \text{if } g(\mathbf{x}) < 0 \quad \text{contact} \end{cases}$$

where ϵ is the penalty parameter.

- Stationary point must satisfy

$$\nabla F_p(\mathbf{x}) = \nabla F(\mathbf{x}) + 2\epsilon \langle -g(\mathbf{x}) \rangle \nabla g(\mathbf{x}) = 0$$

- Solution **tends** to the precise solution as $\epsilon \rightarrow \infty$

■ Lagrange multipliers method

■ Augmented Lagrangian method

Macaulay brackets $\langle x \rangle = \begin{cases} x, & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases} = \max\{x, 0\} = \text{ReLU}(x)$

Optimization methods

Functional to be minimized $F(\mathbf{x})$ under constraint $g(\mathbf{x}) \geq 0$

■ Penalty method $F_p(\mathbf{x}) = F(\mathbf{x}) + \epsilon \langle -g(\mathbf{x}) \rangle^2$

■ Lagrange multipliers method

- New functional called **Lagrangian**

$$\mathcal{L}(\mathbf{x}, \lambda) = F(\mathbf{x}) + \lambda g(\mathbf{x})$$

- Saddle point problem

$$\min_{\mathbf{x}} \max_{\lambda} \{ \mathcal{L}(\mathbf{x}, \lambda) \} \longrightarrow \mathbf{x}^* \longleftarrow \min_{g(\mathbf{x}) \geq 0} \{ F(\mathbf{x}) \}$$

- Stationary point

$$\nabla_{\mathbf{x}, \lambda} \mathcal{L} = \begin{bmatrix} \nabla_{\mathbf{x}} F(\mathbf{x}) + \lambda \nabla_{\mathbf{x}} g(\mathbf{x}) \\ g(\mathbf{x}) \end{bmatrix} = 0 \quad \text{need to verify } \lambda \leq 0$$

■ Augmented Lagrangian method

Macaulay brackets $\langle x \rangle = \begin{cases} x, & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases} = \max\{x, 0\} = \text{ReLU}(x)$

Optimization methods

Functional to be minimized $F(\mathbf{x})$ under constraint $g(\mathbf{x}) \geq 0$

■ Penalty method $F_p(\mathbf{x}) = F(\mathbf{x}) + \epsilon \langle -g(\mathbf{x}) \rangle^2$

■ Lagrange multipliers method $\mathcal{L}(\mathbf{x}, \lambda) = F(\mathbf{x}) + \lambda g(\mathbf{x})$

■ Augmented Lagrangian method

[Hestnes 1969], [Powell 1969], [Glowinski & Le Tallec 1989], [Alart & Curnier 1991], [Simo & Laursen 1992]

- New functional, augmented Lagrangian

$$\mathcal{L}_a(\mathbf{x}, \lambda) = F(\mathbf{x}) + \begin{cases} \lambda g(\mathbf{x}) + \epsilon g^2(\mathbf{x}), & \text{if } \lambda + 2\epsilon g(\mathbf{x}) \leq 0, \text{ contact} \\ -\frac{1}{4\epsilon} \lambda^2, & \text{if } \lambda + 2\epsilon g(\mathbf{x}) > 0, \text{ non-contact} \end{cases}$$

- Stationary point

$$\nabla_{\mathbf{x}, \lambda} \mathcal{L}_a = \begin{cases} \begin{bmatrix} \nabla_{\mathbf{x}} F(\mathbf{x}) + \lambda \nabla_{\mathbf{x}} g(\mathbf{x}) + 2\epsilon g(\mathbf{x}) \nabla g(\mathbf{x}) \\ g(\mathbf{x}) \end{bmatrix} = 0, & \text{if contact} \\ \begin{bmatrix} \nabla_{\mathbf{x}} F(\mathbf{x}) \\ -\frac{\lambda}{\epsilon} \end{bmatrix} = 0, & \text{if non-contact} \end{cases}$$

Macaulay brackets $\langle x \rangle = \begin{cases} x, & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases} = \max\{x, 0\} = \text{ReLU}(x)$



Uzawa algorithm

Optimization methods

Functional to be minimized $F(\mathbf{x})$ under constraint $g(\mathbf{x}) \geq 0$

■ Penalty method $F_p(\mathbf{x}) = F(\mathbf{x}) + \epsilon \langle -g(\mathbf{x}) \rangle^2$

■ Lagrange multipliers method $\mathcal{L}(\mathbf{x}, \lambda) = F(\mathbf{x}) + \lambda g(\mathbf{x})$

■ **Augmented Lagrangian method: Uzawa algorithm**

Iterative procedure for λ update:

$$\mathcal{L}_a(\mathbf{x}; \lambda_i) = F(\mathbf{x}) + \lambda_i g(\mathbf{x}) + \epsilon g^2(\mathbf{x}), \text{ if } \lambda_i + 2\epsilon g(\mathbf{x}) \leq 0$$

$$\mathcal{L}_a(\mathbf{x}; \lambda_i) = F(\mathbf{x}) + (\lambda_i + \epsilon g(\mathbf{x})) g(\mathbf{x}), \text{ if } \lambda_i + 2\epsilon g(\mathbf{x}) > 0$$

$$\lambda_{i+1} = \lambda_i + \epsilon g(\mathbf{x})$$

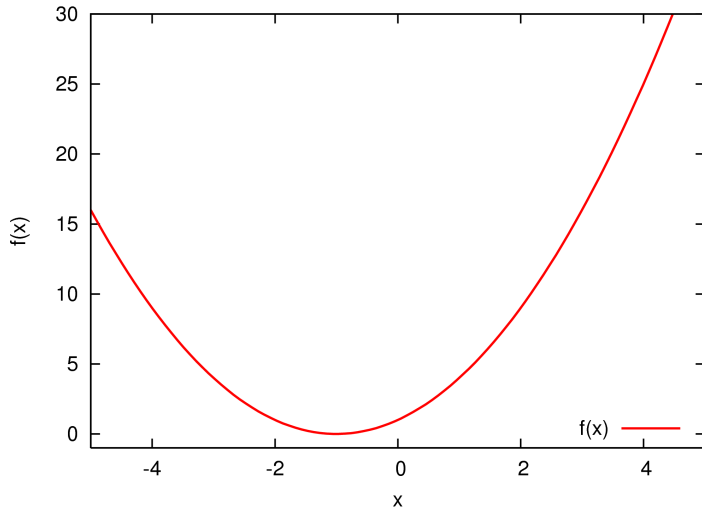
$$\frac{\partial \mathcal{L}_a}{\partial \mathbf{x}} = \frac{\partial F}{\partial \mathbf{x}} + (\lambda_i + 2\epsilon g(\mathbf{x})) \frac{\partial g}{\partial \mathbf{x}} = 0$$

Macaulay brackets $\langle x \rangle = \begin{cases} x, & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases} = \max\{x, 0\} = \text{ReLU}(x)$



Uzawa algorithm

Optimization methods: example

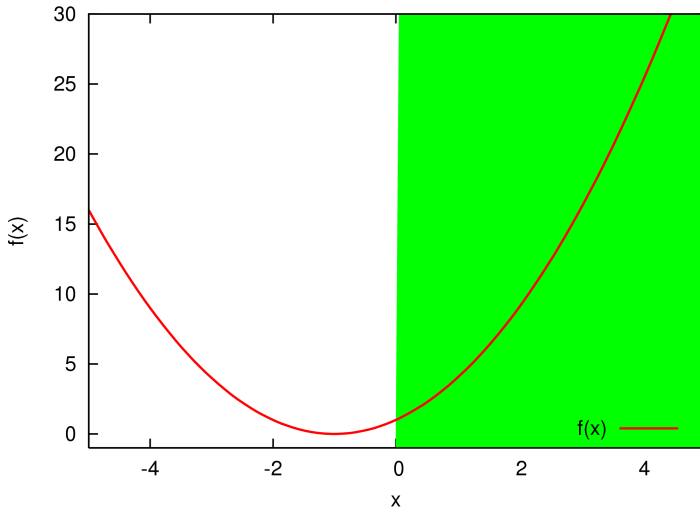


Functional: $f(x) = x^2 + 2x + 1$

Constraint: $g(x) = x \geq 0$

Solution : $x^* = 0$

Optimization methods: example



Functional : $f(x) = x^2 + 2x + 1$

Constraint: $g(x) = x \geq 0$

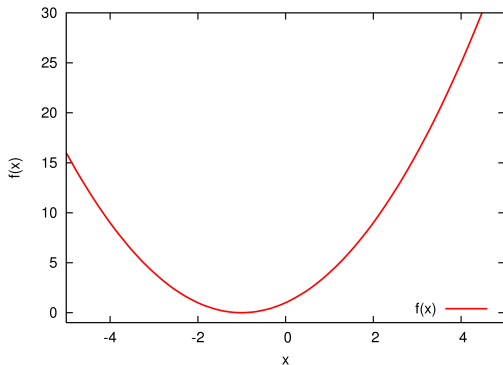
Solution : $x^* = 0$

Penalty method: example

$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

■ Penalty method

$$F_p(x) = F(x) + \epsilon \langle -g(x) \rangle^2$$

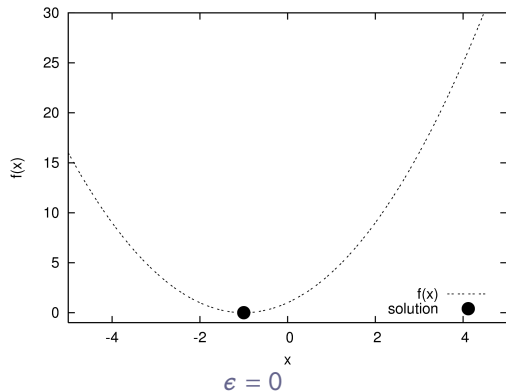
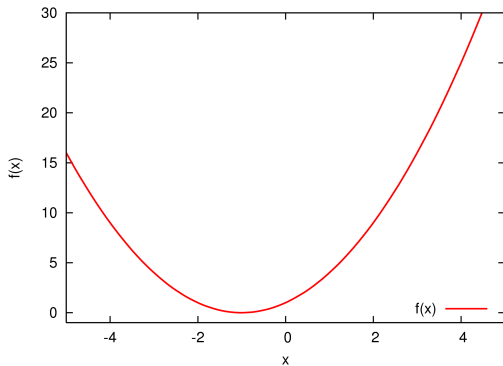


Penalty method: example

$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

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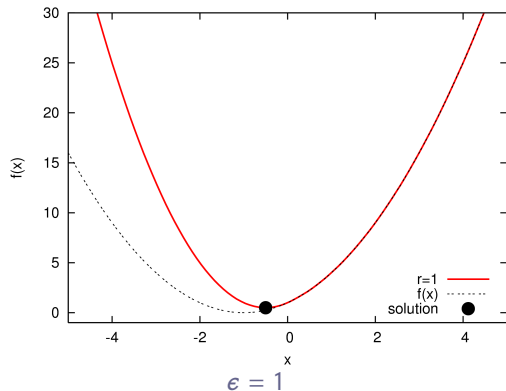
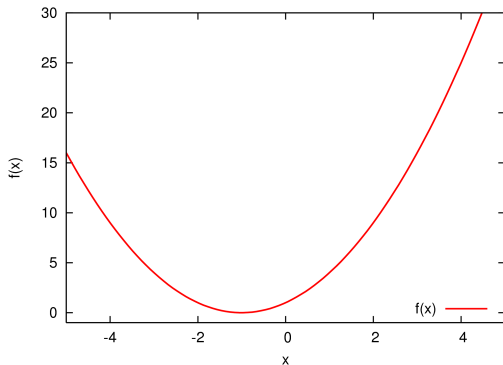


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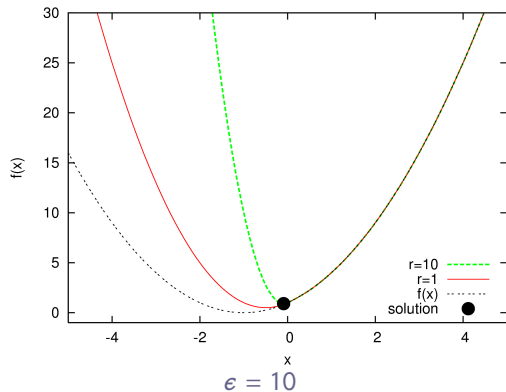
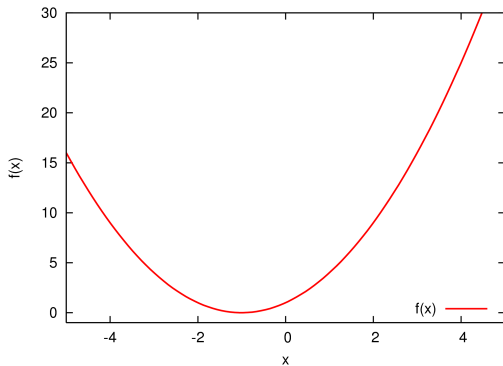


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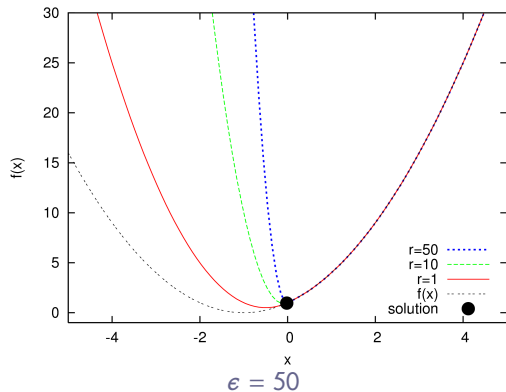
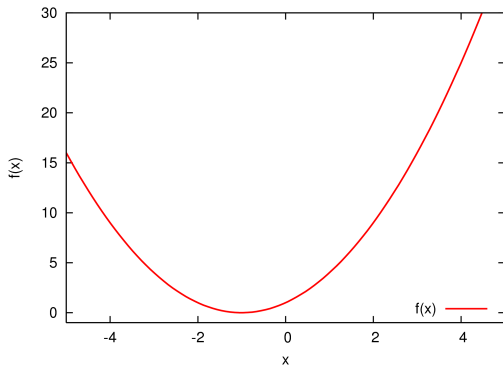


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Penalty method: example

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■ Penalty method

$$F_p(x) = F(x) + \epsilon \langle -g(x) \rangle^2$$

Advantages 😊

- simple physical interpretation
- simple implementation
- no additional degrees of freedom
- “mathematically” smooth functional

Drawbacks 😞

- practically non-smooth functional
- solution is not exact:
 - too small penalty $\rightarrow$ large penetration
 - too large penalty $\rightarrow$ ill-conditioning of the tangent matrix
- user has to choose penalty ϵ properly or automatically and/or adapt during convergence

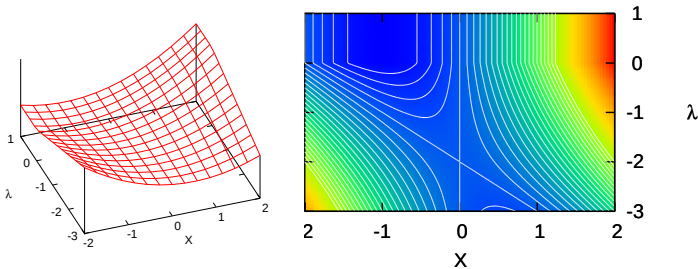
Lagrange multipliers method: example

$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

■ Lagrange multipliers method

$$\mathcal{L}(x, \lambda) = F(x) + \lambda g(x) \rightarrow \text{Saddle point} \rightarrow \min_x \max_{\lambda} \mathcal{L}(x, \lambda)$$

Need to check that $\lambda \leq 0$



Lagrange multipliers method: example

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Need to check that $\lambda \leq 0$

Advantages 😊

- exact solution
- no adjustable parameters

Drawbacks 😞

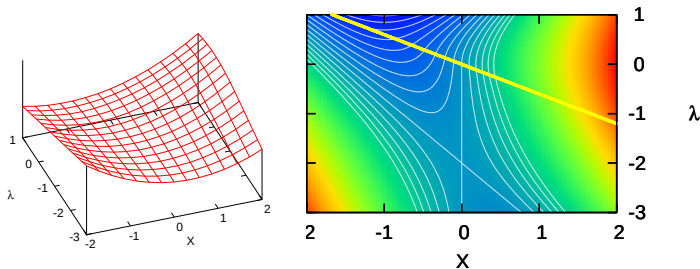
- Lagrangian is not smooth
- additional degrees of freedom
- not fully unconstrained: $\lambda \leq 0$

Augmented Lagrangian method: example

$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

■ Augmented Lagrangian method

$$\mathcal{L}_a(x, \lambda) = F(x) + \begin{cases} \lambda g(x) + \epsilon g^2(x), & \text{if } \lambda + 2\epsilon g(x) \leq 0, \text{ contact} \\ -\frac{1}{4\epsilon} \lambda^2, & \text{if } \lambda + 2\epsilon g(x) > 0, \text{ non-contact} \end{cases}$$



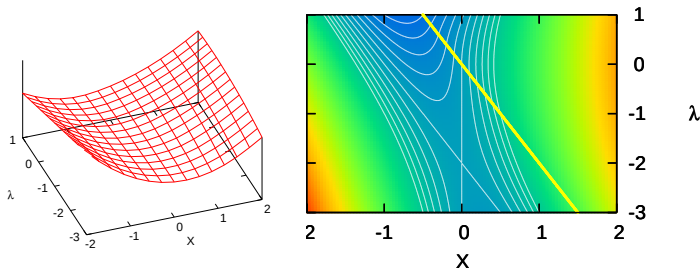
Yellow line separates contact and non-contact regions

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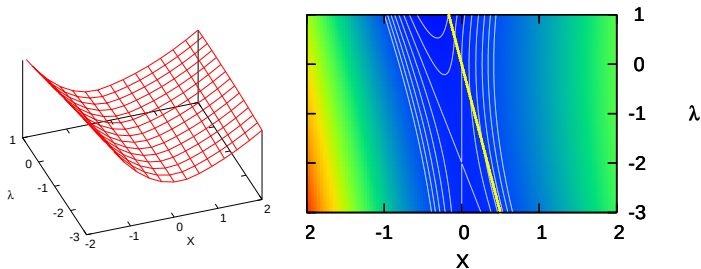
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Advantages 😊

- exact solution
- smoother functional (!)
- fully unconstrained

Drawbacks 😞

- additional degrees of freedom
- quite sensitive to parameter ϵ
- need to adjust ϵ during convergence

Augmented Lagrangian with Uzawa algorithm

■ Augmented Lagrangian method

$$\mathcal{L}_a(\mathbf{x}; \lambda) = F(\mathbf{x}) + \boxed{\lambda g(\mathbf{x})} + \boxed{\epsilon g^2(\mathbf{x})}, \quad \text{if } \lambda + 2\epsilon g(\mathbf{x}) \leq 0, \text{ contact}$$

Fix $\lambda = \lambda_0$

$$\mathcal{L}_a(\mathbf{x}, \lambda) = F(\mathbf{x}) + \lambda_0 g(\mathbf{x}) + \epsilon g^2(\mathbf{x}), \quad \text{if } \lambda_0 + 2\epsilon g(\mathbf{x}) \leq 0$$

Converge with respect to x

Augmented Lagrangian with Uzawa algorithm

■ Augmented Lagrangian method

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Fix $\lambda = \lambda_0$

$$\mathcal{L}_a(\mathbf{x}, \lambda) = F(\mathbf{x}) + [\lambda_0 + \epsilon g(\mathbf{x})] g(\mathbf{x}), \text{ if } \lambda_0 + 2\epsilon g(\mathbf{x}) \leq 0$$

Converge with respect to x and update $\lambda_{i+1} = \lambda_i + \epsilon g(\mathbf{x})$

Augmented Lagrangian with Uzawa algorithm

■ Augmented Lagrangian method

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Fix $\lambda = \lambda_0$

Converge with respect to x and update $\lambda_{i+1} = \lambda_i + \epsilon g(\mathbf{x})$

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Application to contact problems: weak form

$$\int_{\Omega} \underline{\underline{\sigma}} \cdot \delta \nabla \underline{\underline{u}} d\Omega + \underbrace{\int_{\Gamma_c^1} C}_{\text{Contact term}} d\Gamma_c^1 = \int_{\Gamma_f} \underline{\underline{\sigma}}_0 \cdot \delta \underline{\underline{u}} d\Gamma + \int_{\Omega} \underline{\underline{f}}_v \cdot \delta \underline{\underline{u}} d\Omega,$$

$$\underline{\underline{u}} \in \mathbb{L}, \delta \underline{\underline{u}} \in \mathbb{K}, \quad \mathbb{L} = \{ \underline{\underline{u}} \in H^1(\Omega) \mid \underline{\underline{u}} = \underline{\underline{u}}_0 \text{ on } \Gamma_u \}, \quad \mathbb{K} = \{ \delta \underline{\underline{u}} \in H^1(\Omega) \mid \delta \underline{\underline{u}} = 0 \text{ on } \Gamma_u \}$$

■ Penalty method

$$\text{Pressure: } \sigma_n = \epsilon g_n, \text{ Shear: } \underline{\underline{\sigma}}_t = \begin{cases} \epsilon \underline{\underline{g}}_t, & \text{if stick } |\sigma_t| < \mu |\sigma_n| \\ \mu \epsilon g_n \delta \underline{\underline{g}}_t / |\delta \underline{\underline{g}}_t|, & \text{if slip } |\sigma_t| = \mu |\sigma_n| \end{cases}$$

Contact term

$$C = C(g_n, \underline{\underline{g}}_t, \delta g_n, \delta \underline{\underline{g}}_t) = \sigma_n \delta g_n + \underline{\underline{\sigma}}_t \cdot \delta \underline{\underline{g}}_t$$

Application to contact problems: weak form

$$\int_{\Omega} \underline{\underline{\sigma}} \cdot \cdot \delta \nabla \underline{\underline{u}} d\Omega + \underbrace{\int_{\Gamma_c^1} \boxed{C}}_{\text{Contact term}} d\Gamma_c^1 = \int_{\Gamma_f} \underline{\underline{\sigma}}_0 \cdot \delta \underline{\underline{u}} d\Gamma + \int_{\Omega} \underline{\underline{f}}_v \cdot \delta \underline{\underline{u}} d\Omega,$$

$$\underline{\underline{u}} \in \mathbb{L}, \delta \underline{\underline{u}} \in \mathbb{K}, \quad \mathbb{L} = \{ \underline{\underline{u}} \in \mathbb{H}^1(\Omega) \mid \underline{\underline{u}} = \underline{\underline{u}}_0 \text{ on } \Gamma_u \}, \quad \mathbb{K} = \{ \delta \underline{\underline{u}} \in \mathbb{H}^1(\Omega) \mid \delta \underline{\underline{u}} = 0 \text{ on } \Gamma_u \}$$

■ Augmented Lagrangian method

Contact term

$$C = C(g_n, \underline{\underline{g}}_t, \lambda_n, \underline{\underline{\lambda}}_t, \delta g_n, \delta \underline{\underline{g}}_t, \delta \lambda_n, \delta \underline{\underline{\lambda}}_t)$$

$$C = \begin{cases} -\frac{1}{\epsilon} (\lambda_n \delta \lambda_n - \underline{\underline{\lambda}}_t \cdot \delta \underline{\underline{\lambda}}_t), & \text{if non-contact } \lambda_n + \epsilon g_n \geq 0 \\ \hat{\lambda}_n \delta g_n + g_n \delta \lambda_n + \underline{\underline{\hat{\lambda}}}_t \cdot \delta \underline{\underline{g}}_t + \underline{\underline{g}}_t \cdot \delta \underline{\underline{\hat{\lambda}}}_t, & \text{if stick } |\underline{\underline{\hat{\lambda}}}_t| \leq \mu |\hat{\sigma}_n| \\ \hat{\lambda}_n \delta g_n + g_n \delta \lambda_n + \mu \hat{\sigma}_n - \mu \hat{\sigma}_n \frac{\underline{\underline{\hat{\lambda}}}_t}{|\underline{\underline{\hat{\lambda}}}_t|} \cdot \delta \underline{\underline{g}}_t - \frac{1}{\epsilon} \left(\lambda_t + \mu \hat{\sigma}_n \frac{\underline{\underline{\hat{\lambda}}}_t}{|\underline{\underline{\hat{\lambda}}}_t|} \right) \cdot \delta \underline{\underline{\lambda}}_t, & \text{if slip } |\underline{\underline{\hat{\lambda}}}_t| \geq \mu |\hat{\sigma}_n| \end{cases}$$

where $\hat{\lambda}_n = \lambda_n + \epsilon g_n$ and $\underline{\underline{\hat{\lambda}}}_t = \underline{\underline{\lambda}}_t + \epsilon \underline{\underline{g}}_t$.

Application to contact problems: linearization

- Non-linear equation

$$R(\underline{u}, \underline{f}) = 0$$

- Contains $\delta g_n, \delta \underline{g}_t$
- Use Newton-Raphson method
- Initial state at step i

$$R(\underline{u}^i, \underline{f}^i) = 0$$

- Should be also satisfied at step $i + 1$

$$R(\underline{u}^{i+1}, \underline{f}^{i+1}) = R(\underline{u}^i + \delta \underline{u}, \underline{f}^{i+1}) = 0$$

- Linearize

$$R(\underline{u}^i + \delta \underline{u}, \underline{f}^{i+1}) = R(\underline{u}^i, \underline{f}^{i+1}) + \frac{\partial R(\underline{u})}{\partial \underline{u}} \delta \underline{u} = 0$$

- Finally

$$\delta \underline{u} = - \underbrace{\left[\frac{\partial R(\underline{u})}{\partial \underline{u}} \right]^{-1}}_{\text{contains } \Delta \delta g_n, \Delta \delta \underline{g}_t} R(\underline{u}^i)$$

- NB: Contact problem does not satisfy conditions of Kantorovich theorem on the convergence of Newton's method.

Variation of geometrical quantities

Normal gap

- First variation enters in the residual vector:

$$\delta g_n = \underline{n} \cdot (\delta \underline{r}_s - \delta \underline{\rho})$$

- Second variation enters in the tangent matrix:

$$\begin{aligned} \Delta \delta g_n = & -\underline{n} \cdot \left(\delta \frac{\partial \underline{\rho}}{\partial \underline{\xi}} \Delta \underline{\xi} + \Delta \frac{\partial \underline{\rho}}{\partial \underline{\xi}} \delta \underline{\xi} \right) - \Delta \underline{\xi}^T \underline{\mathbb{H}} \delta \underline{\xi} + \\ & + g_n \left(\Delta \underline{\xi}^T \underline{\mathbb{H}} + \underline{n} \cdot \Delta \frac{\partial \underline{\rho}}{\partial \underline{\xi}} \right) \bar{\underline{A}} \left(\underline{n} \cdot \delta \frac{\partial \underline{\rho}}{\partial \underline{\xi}} + \underline{\mathbb{H}} \delta \underline{\xi} \right) \end{aligned}$$

Variation of geometrical quantities

Convective coordinate of the projection

- First variation enters in the residual vector:

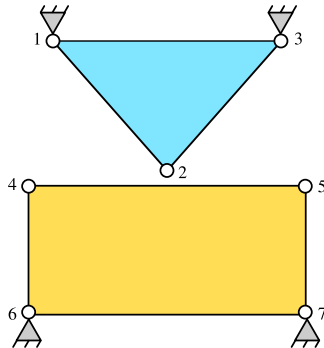
$$\delta \xi = \left[\underline{\underline{A}} - g_n \underline{\underline{H}} \right]^{-1} \left(\frac{\partial \underline{\rho}}{\partial \xi} \cdot (\delta \underline{r}_s - \delta \underline{\rho}) + g_n \underline{n} \cdot \delta \frac{\partial \underline{\rho}}{\partial \xi} \right)$$

- Second variation enters in the tangent matrix:

$$\begin{aligned} \Delta \delta \xi = & (g_n \underline{\underline{H}} - \underline{\underline{A}})^{-1} \left\{ \frac{\partial \underline{\rho}}{\partial \xi} \cdot \left(\delta \frac{\partial \underline{\rho}^T}{\partial \xi} \Delta \xi + \Delta \frac{\partial \underline{\rho}^T}{\partial \xi} \delta \xi \right) + \Delta \xi^T \left(\frac{\partial \underline{\rho}}{\partial \xi} \cdot \frac{\partial^2 \underline{\rho}}{\partial \xi^2} \right) \delta \xi - \right. \\ & - g_n \underline{n} \cdot \left(\delta \frac{\partial^2 \underline{\rho}}{\partial \xi^2} \Delta \xi + \Delta \frac{\partial^2 \underline{\rho}}{\partial \xi^2} \delta \xi \right) - g_n \Delta \xi^T \left(\underline{n} \cdot \frac{\partial^3 \underline{\rho}}{\partial \xi^3} \right) \delta \xi + \\ & + \left[g_n \left(\delta \frac{\partial \underline{\rho}}{\partial \xi} + \frac{\partial^2 \underline{\rho}}{\partial \xi^2} \delta \xi \right) \cdot \frac{\partial \underline{\rho}^T}{\partial \xi} \bar{\underline{\underline{A}}} - \delta g_n \underline{\underline{I}} \right] \left(\underline{n} \cdot \Delta \frac{\partial \underline{\rho}}{\partial \xi} + \underline{\underline{H}} \Delta \xi \right) + \\ & + \left[g_n \left(\Delta \frac{\partial \underline{\rho}}{\partial \xi} + \frac{\partial^2 \underline{\rho}}{\partial \xi^2} \Delta \xi \right) \cdot \frac{\partial \underline{\rho}^T}{\partial \xi} \bar{\underline{\underline{A}}} - \Delta g_n \underline{\underline{I}} \right] \left(\underline{n} \cdot \delta \frac{\partial \underline{\rho}}{\partial \xi} + \underline{\underline{H}} \delta \xi \right) \Big\} \end{aligned}$$

Example

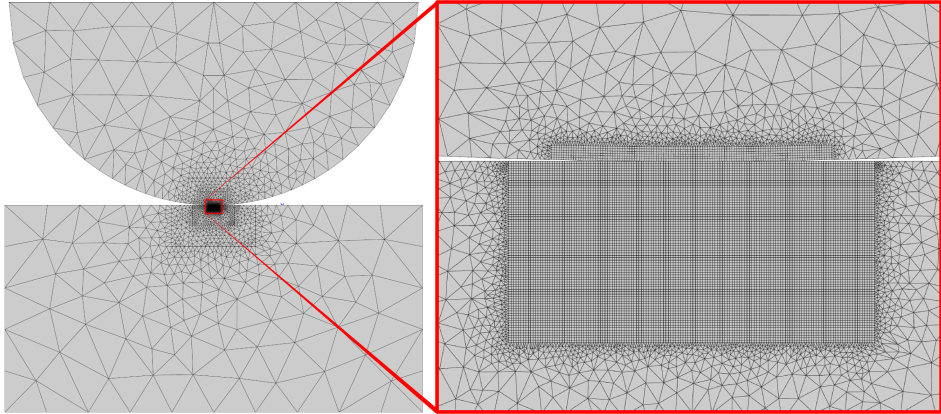
- Use penalty method to enforce Dirichlet BC
- Use penalty method to enforce contact constraints
- First, detect contact elements
- Second, construct updated residual vector and tangent matrix



Contact between two elements

Particularities: mesh and convergence

- Strong **mesh refinement** is required
 - especially at **unknown edges** of contact zones

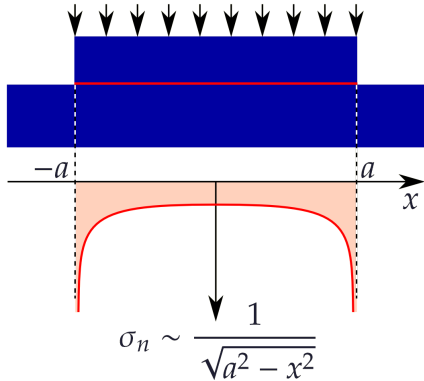


Typical mesh for fretting analysis [L. Sun, H. Proudhon, G. Cailletaud, 2011]

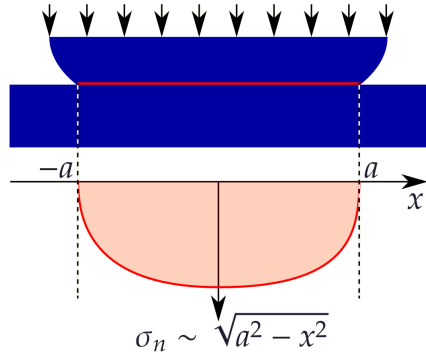
2D ~ 30 000 DoFs, 3D ~ 5 000 000 DoFs

Particularities: mesh and convergence

- Strong **mesh refinement** is required
 - especially at **unknown edges** of contact zones



$$\sigma_n \xrightarrow{x \rightarrow a} -\infty \quad \left| \frac{\partial \sigma_n}{\partial x} \right| \xrightarrow{x \rightarrow a} \infty$$

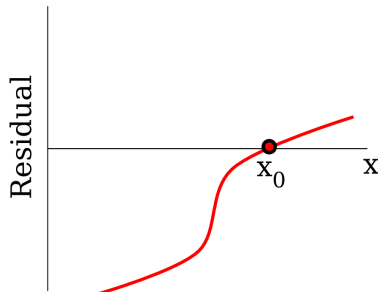


$$\left| \frac{\partial \sigma_n}{\partial x} \right| \xrightarrow{x \rightarrow a} \infty$$

Infinite contact pressure and/or its derivative

Particularities: mesh and convergence

- Strong **mesh refinement** is required
 - especially at **unknown edges** of contact zones
 - **Slow change** of boundary conditions:
 - strong non-linearities of contact / friction problems
 - non-uniqueness of solution for frictional problems
- Infinite looping**

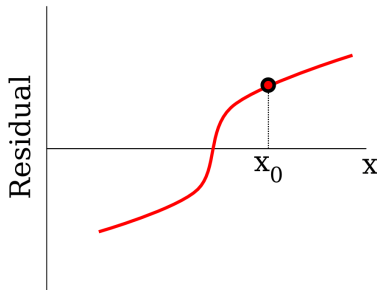


Initial guess $R(x_0, f_0) = 0$

Particularities: mesh and convergence

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Infinite looping

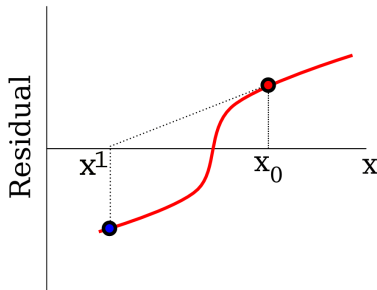


Too rapid change in boundary conditions $R(x_0, f_1) \neq 0$

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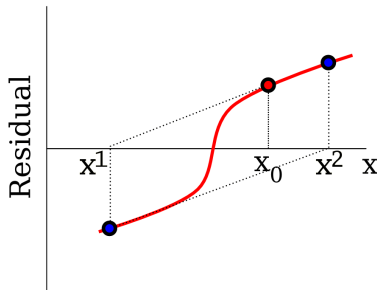


Iterations of Newton-Raphson method $R(x_0, f_1) + \frac{\partial R}{\partial x} \Big|_{x_0} \delta x = 0 \rightarrow \delta x = - \frac{\partial R}{\partial x} \Big|_{x_0}^{-1} R(x_0, f_1) \rightarrow x^1 = x_0 + \delta x$

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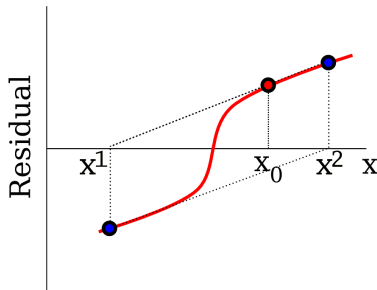


Iterations of Newton-Raphson method $R(x^1, f_1) + \frac{\partial R}{\partial x} \Big|_{x^1} \delta x = 0 \rightarrow \delta x = - \frac{\partial R}{\partial x} \Big|_{x^1}^{-1} R(x^1, f_1) \rightarrow x^2 = x^1 + \delta x$

Particularities: mesh and convergence

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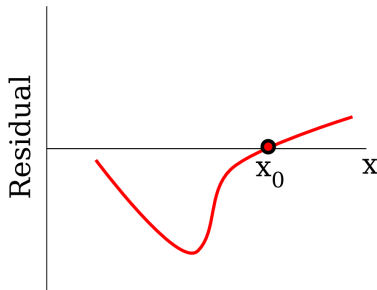
Infinite looping



Infinite looping

Particularities: mesh and convergence

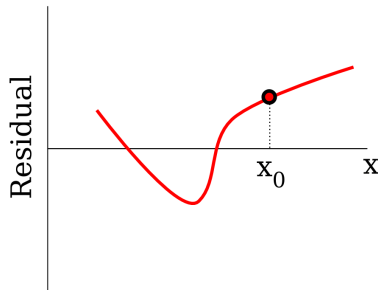
- Strong **mesh refinement** is required
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- Convergence to a “false” solution**



Initial guess $R(x_0, f_0) = 0$

Particularities: mesh and convergence

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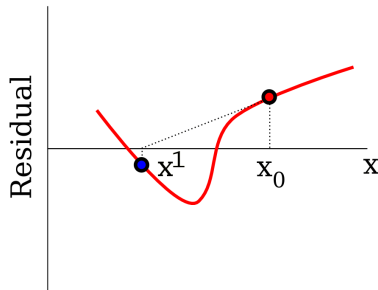


Too rapid change in boundary conditions $R(x_0, f_1) \neq 0$

Particularities: mesh and convergence

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Convergence to a “false” solution

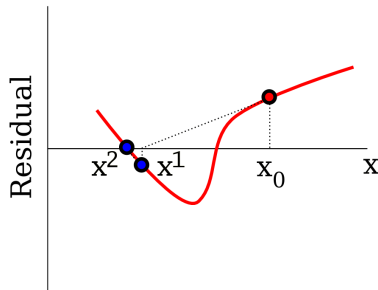


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Convergence to a “false” solution

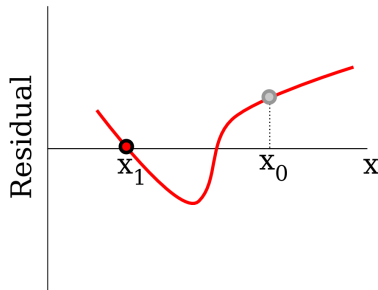


Iterations of Newton-Raphson method $R(x^1, f_1) + \frac{\partial R}{\partial x} \Big|_{x^1} \delta x = 0 \rightarrow \delta x = - \frac{\partial R}{\partial x} \Big|_{x^1}^{-1} R(x^1, f_1) \rightarrow x^2 = x^1 + \delta x$

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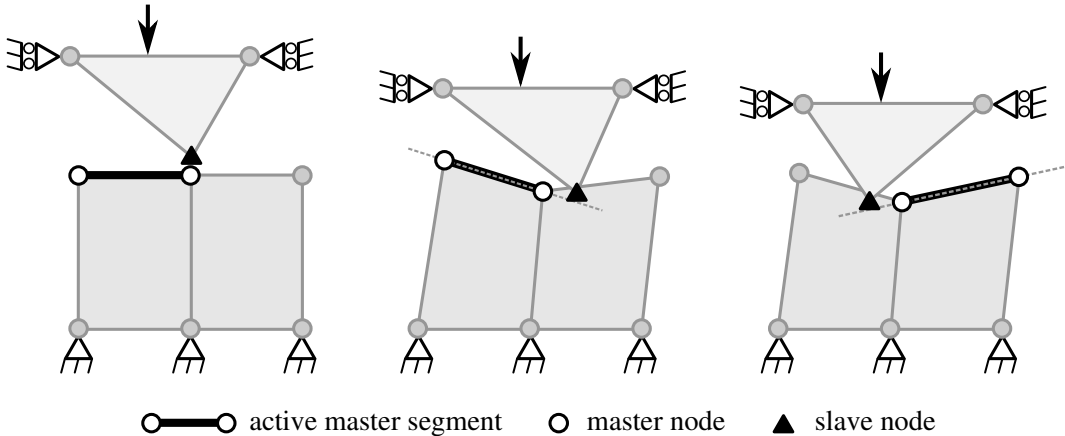
Convergence to a “false” solution



Convergence, but is it a “true” solution ?

Convergence problems: examples

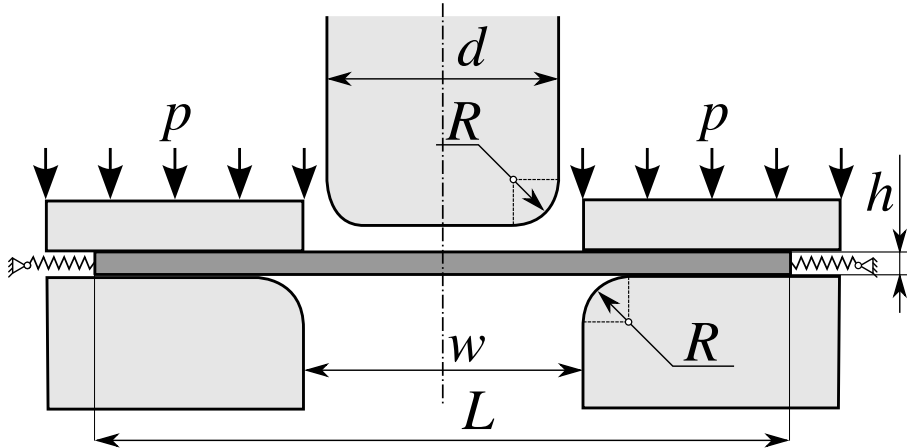
■ Infinite looping, e.g.



- Change of the contact state (contact/non-contact, stick/slip)
- Interplay between stiffness, friction and augmented Lagrangian coefficients^[1]
- Combination of non-linearities (e.g., plasticity+contact)

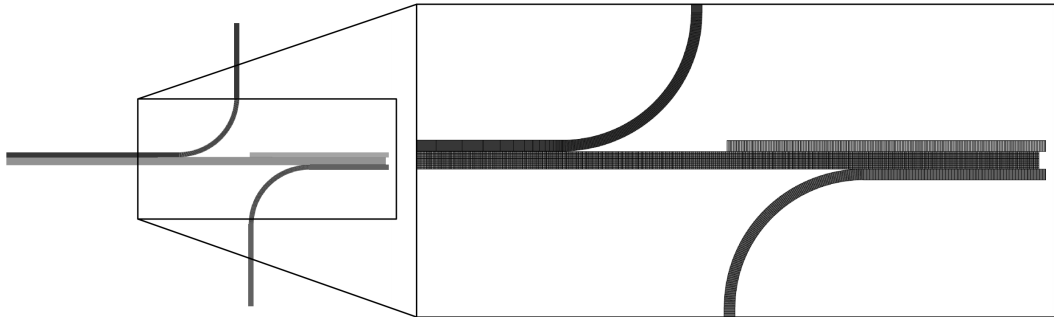
Convergence problems: examples

- Simulation of a deep drawing problem
- Finite strain plasticity + frictional contact



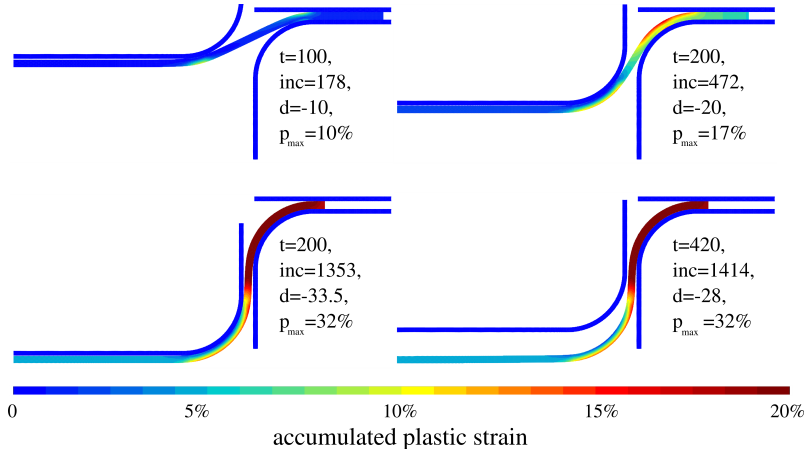
Convergence problems: examples

- Simulation of a deep drawing problem
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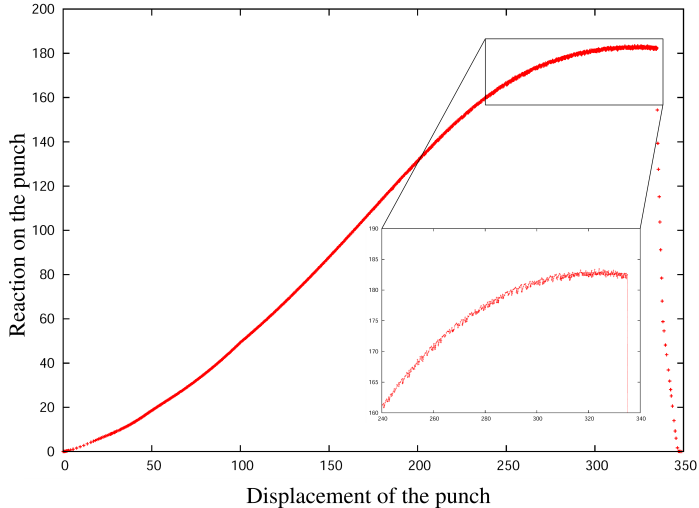
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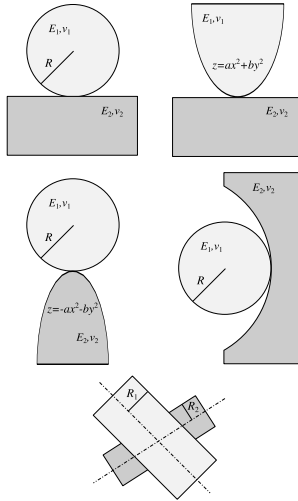
Convergence problems: examples

- Simulation of a deep drawing problem
- Finite strain plasticity + frictional contact



Cylinder-plane frictional contact

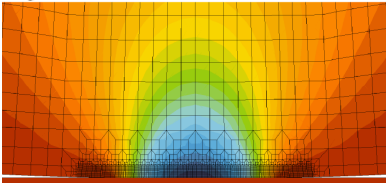
- Non-conservative problem, history of loading is crucial



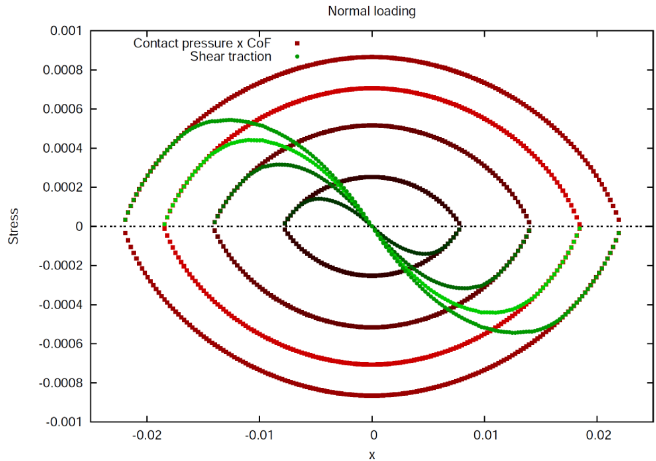
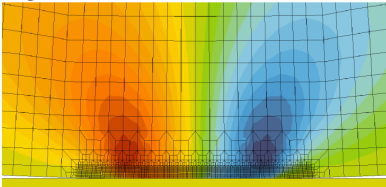
Cylinder-plane frictional contact

- Non-conservative problem, history of loading is crucial

sig22 at maximal normal load



sig12 at maximal normal load

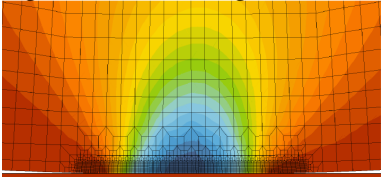


Press in 100 increments, $u_z \sim t^2$

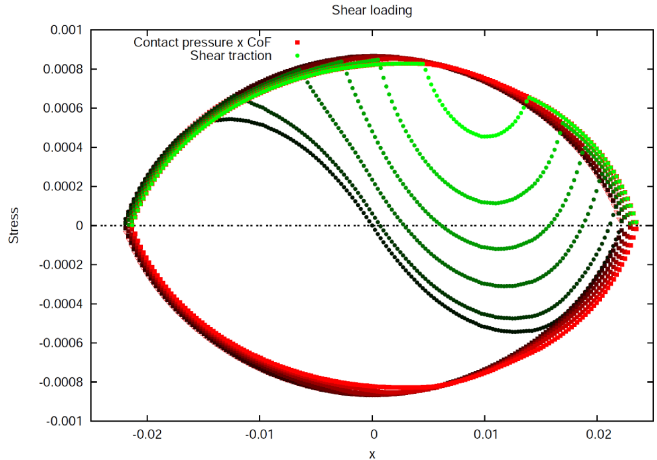
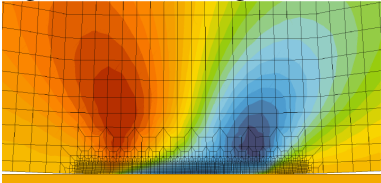
Cylinder-plane frictional contact

- Non-conservative problem, history of loading is crucial

sig22 at maximal tangential load



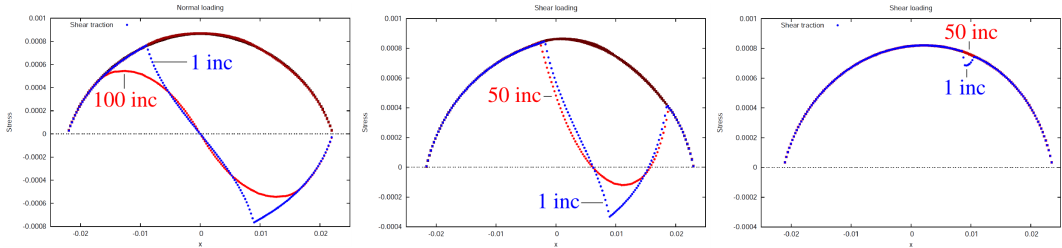
sig12 at maximal tangential load



Shift in 100 increments, $u_z \sim t$

Cylinder-plane frictional contact

- Non-conservative problem, history of loading is crucial

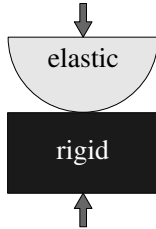


Comparison with: press in 1 increment, shift in 2 increments

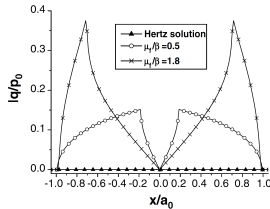
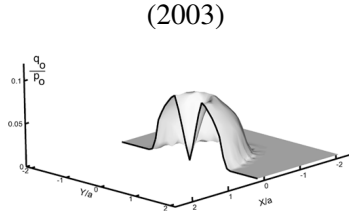
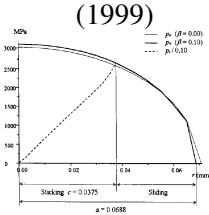
Before sticking, every point of the contact interface has to pass through the slip zone. It is impossible when loaded too fast.

Warning friction!

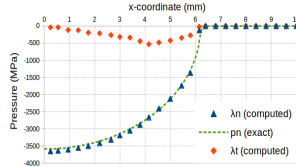
- For dissimilar materials, the *friction matters* even in normal contact
- The problem is thus path-dependent, the B.C. should be changed slowly



Erroneous solutions

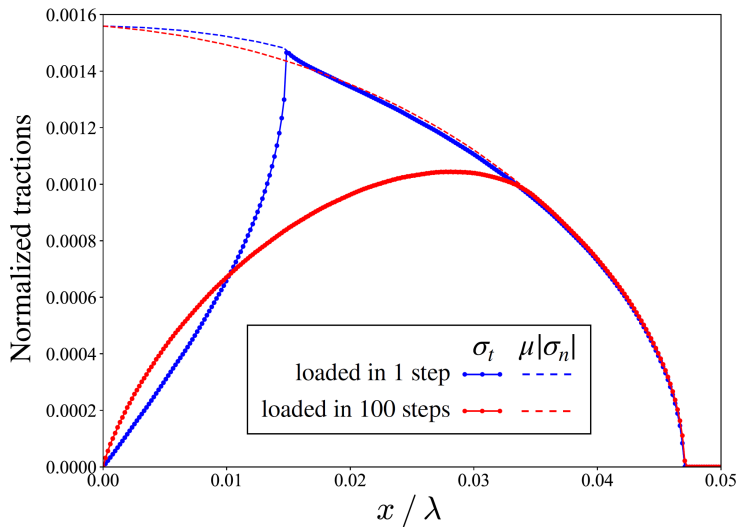


(2008)

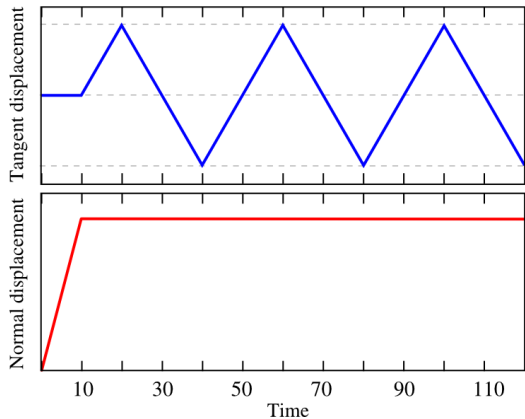
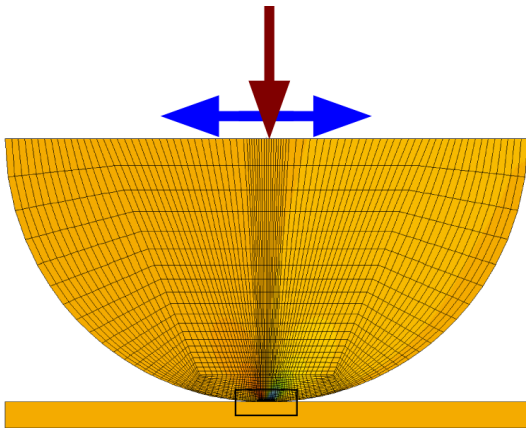


(2017)

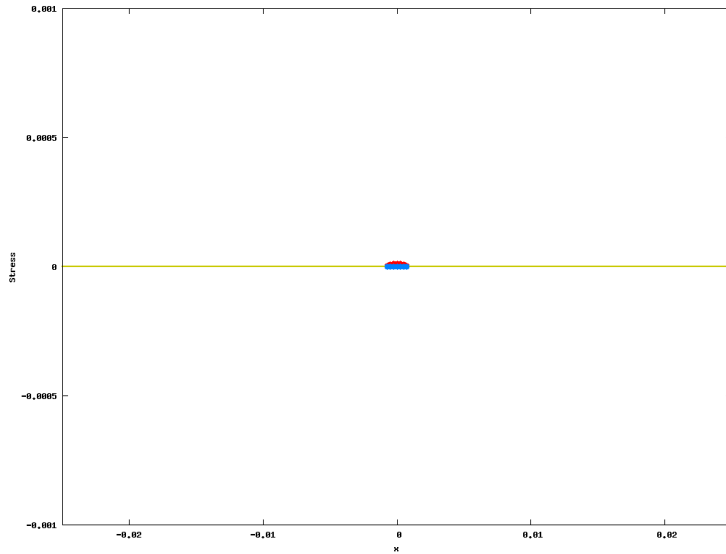
Warning friction!



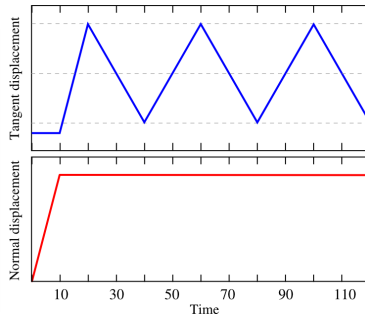
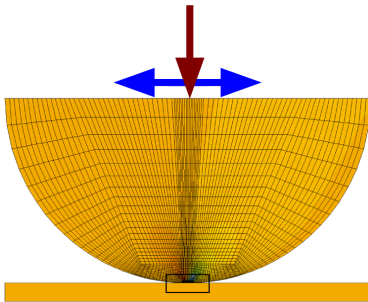
Sphere-plane frictional contact: cycling



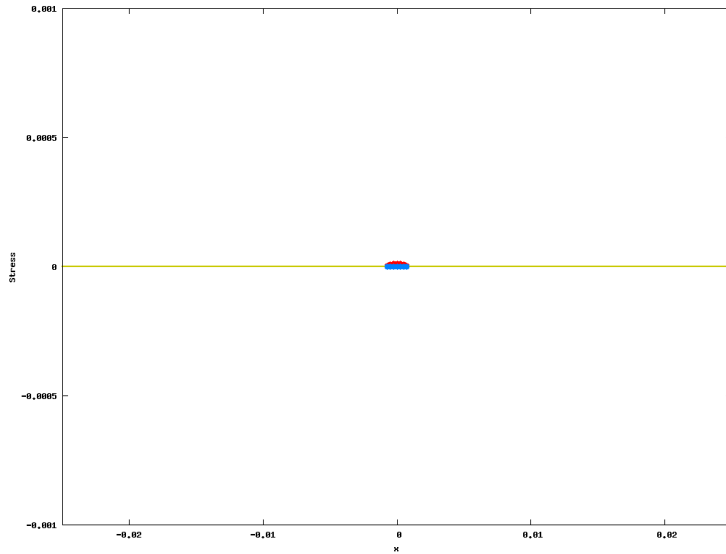
Sphere-plane frictional contact: cycling



Sphere-plane frictional contact: cycling

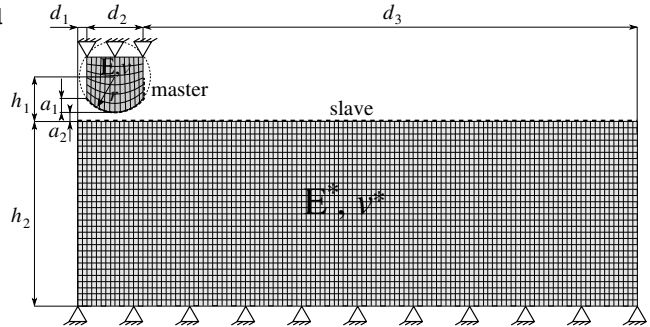


Sphere-plane frictional contact: cycling



Shallow ironing test

- Deformable-on-deformable frictional sliding
- Results obtained by different groups^{1,2,3,4,5,6} differ significantly
- Local and global friction coefficients may differ



[1] Fischer K. A., Wriggers P., "Mortar based frictional contact formulation for higher order interpolations using the moving friction cone", Computer Methods in Applied Mechanics and Engineering, vol. 195, p. 5020-5036, 2006.

[2] Hartmann S., Oliver J., Cante J. C., Weyler R., Hernández J. A., "A contact domain method for large deformation frictional contact problems. Part 2: Numerical aspects", Computer Methods in Applied Mechanics and Engineering, vol. 198, p. 2607-2631, 2009.

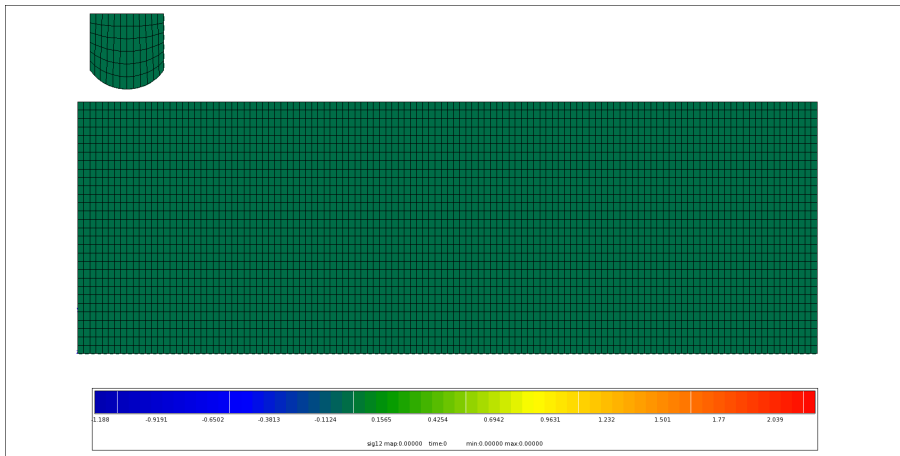
[3] Yastrebov V. A., "Computational contact mechanics: geometry, detection and numerical techniques", Thèse CdM & Onera, 2011.

[4] Kudawoo A. D., "Problèmes industriels de grande dimension en mécanique numérique du contact : performance, fiabilité et robustesse", Thèse @ LMA & LAMSID, 2012.

[5] Poullos K., Renard Y., "A non-symmetric integral approximation of large sliding frictional contact problems of deformable bodies based on ray-tracing", soumis, 2014.

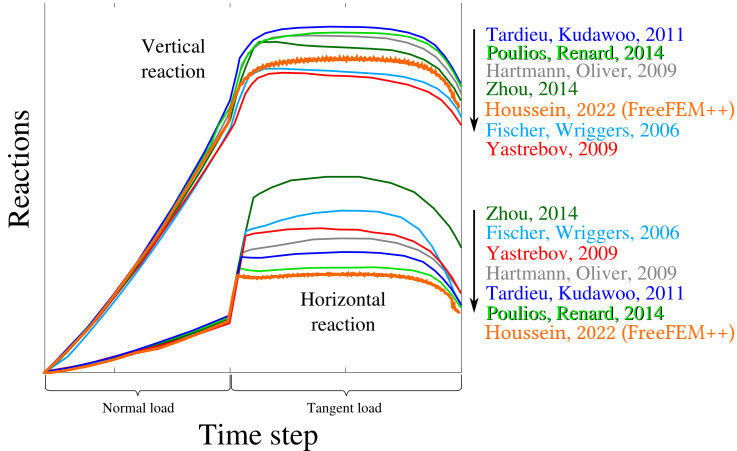
[6] Zhou Lei's blog, <http://kt2008plus.blogspot.de>

Shallow ironing test



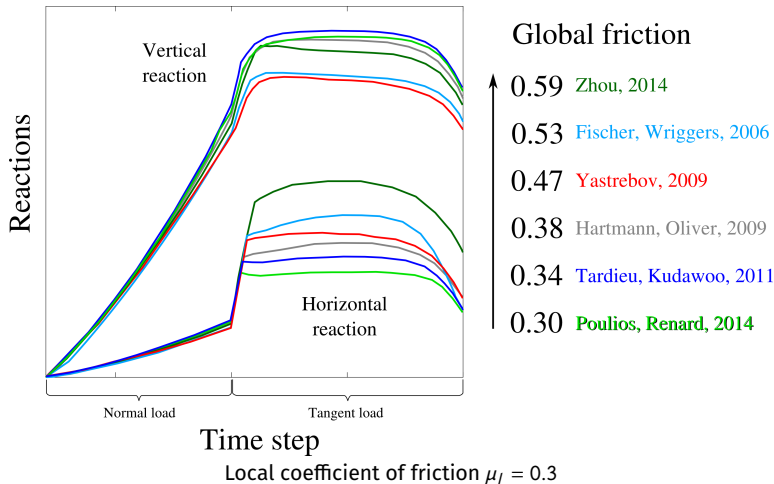
Shallow ironing test

- No agreement between authors
- Dif. authors used dif. meshes (quadrilateral lin./sq., triangular lin.)
- Dif. authors used either finite or infinitesimal strain formulation



Shallow ironing test

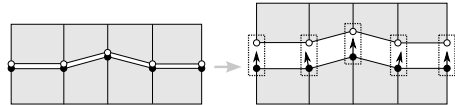
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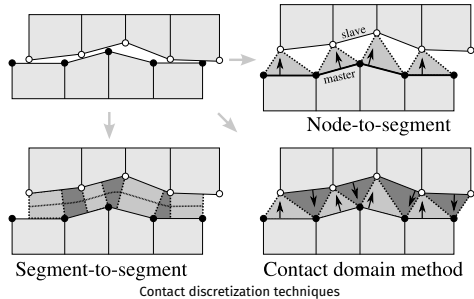
Reading

- It's just a tip of the “Computational Contact Mechanics” iceberg
- Contact discretization and integration
- Smoothing techniques
- Energy conservative methods for dynamics

Infinitesimal deformation / infinitesimal sliding

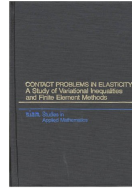


General case



- It's just a tip of the “Computational Contact Mechanics” iceberg
- Contact discretization and integration
- Smoothing techniques
- Energy conservative methods for dynamics

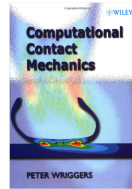
Kikuchi, Oden (1988)



Zhong (1993)



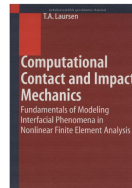
Wriggers (2002)



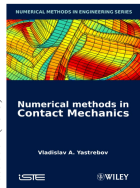
Wriggers, 2nd ed. (2006)



Laursen (2002)



Yastrebov (2013)



$\mathcal{L}_a(x, \lambda)$

Thank you for your attention!

$$\mathcal{L}_a(x, \lambda)$$

Thank you for your attention!

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Contact Mechanics and Elements of Tribology

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