Nonlinear Computational Mechanics

Computational Contact Mechanics

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@ Ecole des Mines de Paris
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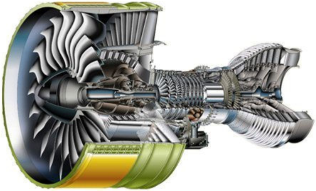
Outline

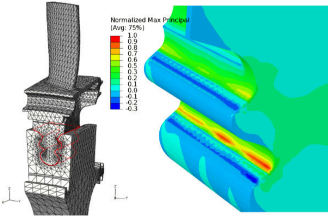
Introduction

Industrial and natural contact problems

1. Assembled parts, e.g. engines



Aircraft's engine GP 7200
`www.safran-group.com`

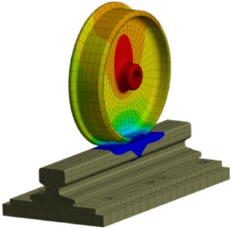


[1] M. W. R. Savage
J. Eng. Gas Turb. Power, 134:012501 (2012)

Industrial and natural contact problems

1. Assembled parts, e.g. engines
2. Railroad contacts

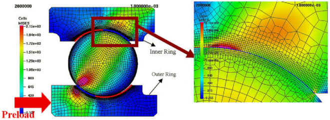
High speed train TGV www.sncf.com



Wilde/ANSYS wildeanalysis.co.uk

1. Assembled parts, e.g. engines
2. Railroad contacts
3. Gears and bearings

Bearings
www.skf.com



[1] F. Massi, J. Rocchi, A. Culla, Y. Berthier Mech. Syst. Signal Pr., 24:1068-1080 (2010)

Industrial and natural contact problems

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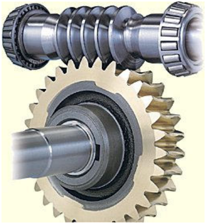
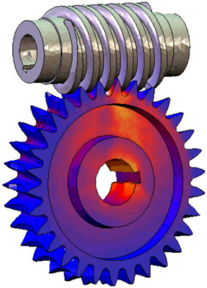


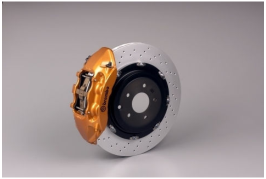
Helical gear www.tpg.com.tw



www.mscsoftware.com

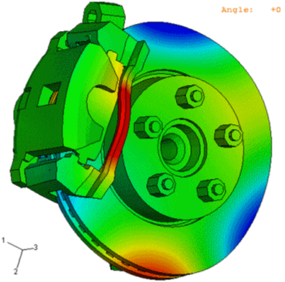
Industrial and natural contact problems

1. Assembled parts, e.g. engines
2. Railroad contacts
3. Gears and bearings
4. Breaking systems



Assembled breaking system

www.brembo.com



www.mechanicalengineeringblog.com

Industrial and natural contact problems

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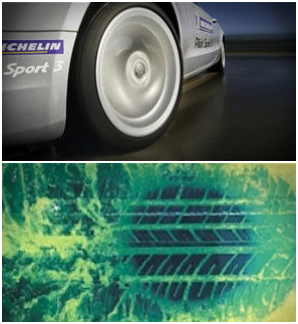
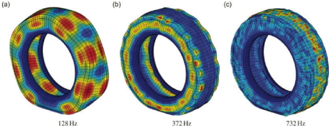
3. Gears and bearings
4. Breaking systems
5. Tire-road contact



Tire-road contact www.michelin.com



[1] M. Brinkmeier, U. Nackenhorst, S. Petersen, O. von Estorff, J. Sound Vib., 309:20-39 (2008)

Industrial and natural contact problems

1. Assembled parts, e.g. engines
2. Railroad contacts
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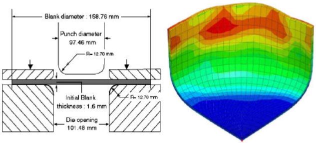
5. Tire-road contact
6. Metal forming

Deep drawing `www.thomasnet.com`



[1] G. Rousselier, F. Barlat, J. W. Yoon
Int. J. Plasticity, 25:2383-2409 (2009)

Industrial and natural contact problems

1. Assembled parts, e.g. engines
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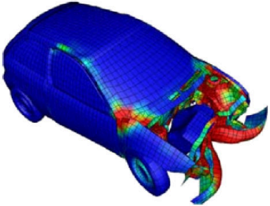
6. Metal forming
7. Crash tests

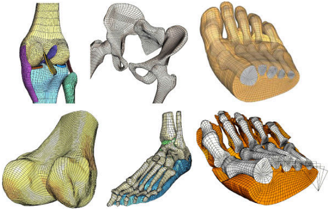
Crash-test www.porsche.com



[1] O. Klyavin, A. Michailov, A. Borovkov
www.fea.ru

1. Assembled parts, e.g. engines
2. Railroad contacts
3. Gears and bearings
4. Breaking systems
5. Tire-road contact
6. Metal forming
7. Crash tests
8. Biomechanics

Human articulations
`www.sportssupplements.net`



J. A. Weiss, University of Utah
Musculoskeletal Research Laboratories

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6. Metal forming
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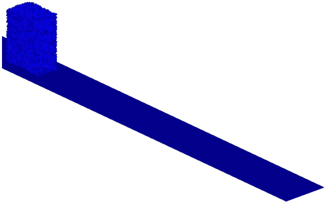
8. Biomechanics
9. Granular materials

Sand dunes `www.en.wikipedia.org`



E. Azema et al, LMGC90

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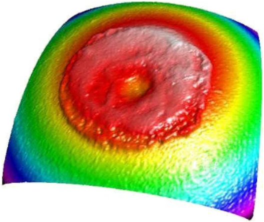
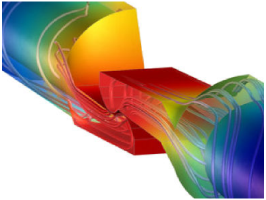
9. Granular materials
10. Electric contacts



Damage at electric contact zone
www.taicaan.com



Simulation of electric current
www.comsol.com

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8. Biomechanics
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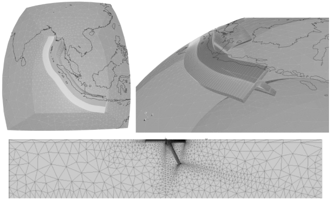
10. Electric contacts
11. Tectonic motions

San-Andreas fault, by M. Rightmire
`www.sciencedude.ocregister.com`



[1] J.D. Garaud, L. Fleitout, G. Cailletaud
Colloque CSMA (2009)

Industrial and natural contact problems

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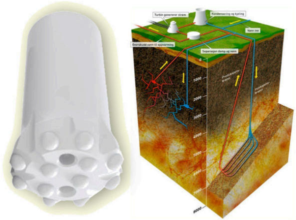
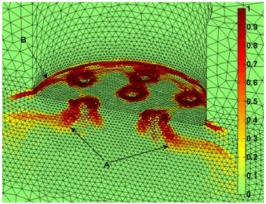
11. Tectonic motions
12. Deep drilling



Drill Bit tool **RobitRocktools**; *extraction of geothermal energy* (**SINTEF, NTNU**)



[1] T. Saksala, Int. J. Numer. Anal. Meth. Geomech. (2012)

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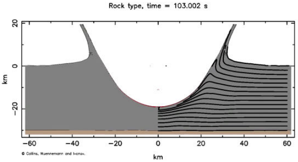
10. Electric contacts
11. Tectonic motions
12. Deep drilling
13. Impact and fragmentation

Impact crater, Arizona
www.MrEclipse.com *et* maps.google.com



Simulation of formation of Copernicus crater
Yue Z., Johnson B. C., et al. Projectile remnants in central peaks of lunar impact craters. Nature Geo 6 (2013)

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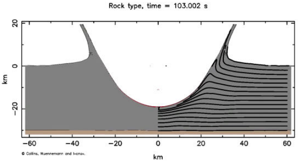
14. etc.

Impact crater, Arizona
www.MrEclipse.com *et* maps.google.com



Simulation of formation of Copernicus crater
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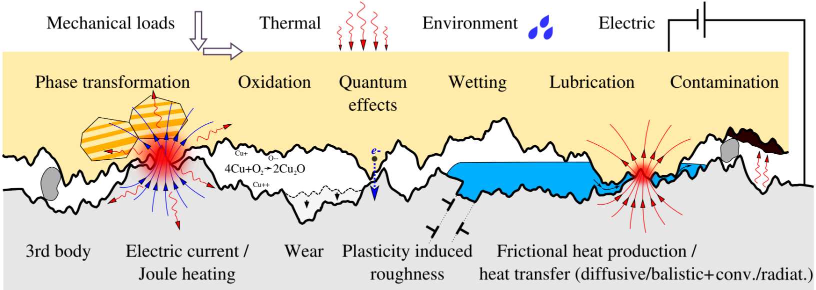
Physical and mathematical complexity

- Contact interface is hard to observe in situ
- Many things happen in the interface
- Strong thermo-mechanical or fluid-solid coupling in sliding
- Mathematical formulation is also non-trivial



Interface between two solids in contact

Basics of Contact and Friction

Notations

Vectors and tensors

• a, α	scalars	• $\underline{a} \cdot \underline{b} = c$	scalar (dot) product
• $\underline{b}$	vectors	• $\underline{a} \times \underline{b} = \underline{c}$	vector (cross) product
• $\underline{\underline{C}}, \underline{\underline{\beta}}$	2nd order tensors	• $\underline{a} \otimes \underline{b} = \underline{\underline{C}}$	tensor product
• $\overset{4}{\underline{\underline{D}}}$	4th order tensors	• $\underline{\underline{A}}^T$	transposition
• $\nabla \underline{a} = \underline{\underline{B}}$	gradient operator	• $\nabla \cdot \underline{a} = c$	divergence operator
• $\nabla \times \underline{a} = \underline{b}$	curl (rot) operator	• $\underline{\underline{I}} = \underline{e}_i \otimes \underline{e}_i$	2nd order identity tensor

Mechanics

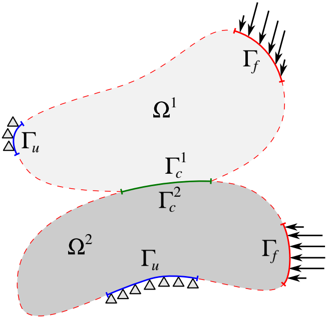
• $\underline{\underline{\sigma}}$	Cauchy stress tensor	• $\underline{\underline{\varepsilon}}$	Small strain tensor
• g, g_n	gap, normal gap	• $\underset{\sim}{\xi}$	position vector in parent space
• ϵ	penalty parameter	• $\underline{n}$	outward unit normal vector
• $\lambda, \lambda_n, \lambda_t$	lagrange multipliers	• $\frac{\partial \underline{\rho}}{\partial \xi_1}, \frac{\partial \underline{\rho}}{\partial \xi_2}$	surface tangent vectors
• $\sigma_n = (\underline{\underline{\sigma}} \cdot \underline{n}) \cdot \underline{n}$	contact pressure	• f, μ	Coefficient of friction

- **Balance of momentum**

$$\begin{cases} \nabla \cdot \underline{\underline{\sigma}} + \underline{f_v} = 0 & \text{in } \Omega_{1,2} \\ \underline{\underline{\sigma}} \cdot \underline{n} = \underline{t}_0 & \text{on } \Gamma_f \\ \underline{u} = \underline{u}_0 & \text{on } \Gamma_u \\ ? & \text{on } \Gamma_c \end{cases}$$

- **Frictionless contact conditions *(intuitive)***

  1. No penetration
  2. No adhesion
  3. No shear transfer



Equilibrium and contact conditions

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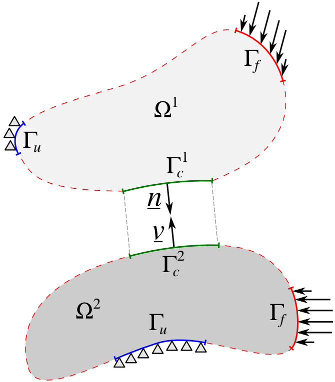

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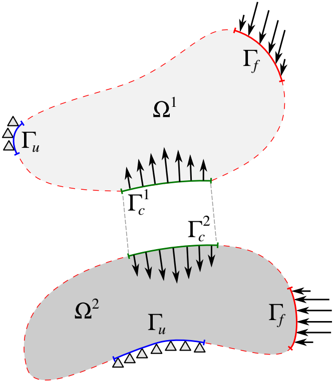

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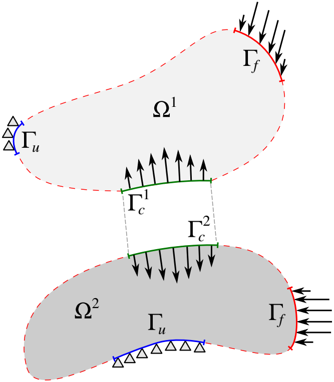

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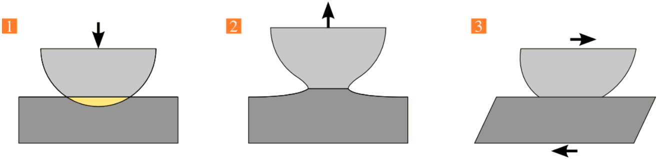

Equilibrium and contact conditions

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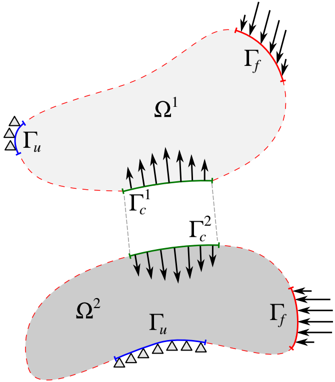
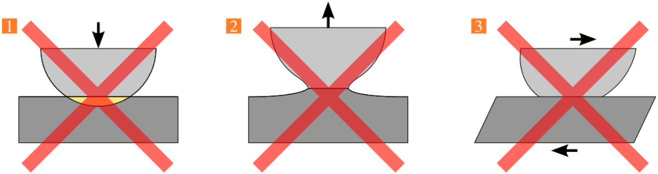
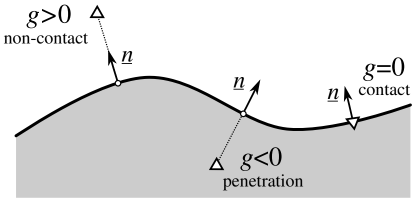




Gap function

- **Gap function g**
  - gap = – penetration
  - asymmetric function
  - defined for
    - separation $g > 0$
    - contact $g = 0$
    - penetration $g < 0$
  - governs normal contact
- **Master and slave split**

  Gap function is determined for all slave points with respect to the master surface



Gap between a slave point and a master surface

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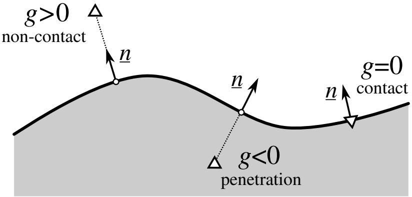
  - governs normal contact
- **Master and slave split**

  Gap function is determined for all slave points with respect to the master surface
- **Normal gap**

$$g_n = \underline{n} \cdot \left[\underline{r}_s - \underline{\rho}(\xi_\pi) \right],$$

$\underline{n}$ is a unit normal vector, $\underline{r}_s$ slave point, $\underline{\rho}(\xi_\pi)$ projection point at master surface



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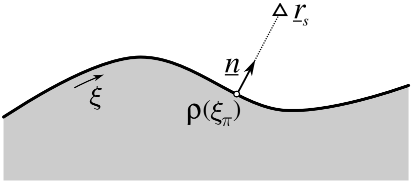
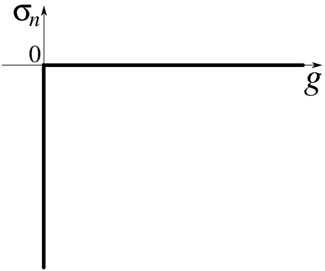


Definition of the normal gap

Consider existence and uniqueness 

- **No penetration**
  Always non-negative gap
  $$g \ge 0$$
- **No adhesion**
  Always non-positive contact pressure
  $$\sigma_n^* \le 0$$
- **Complementary condition**
  Either zero gap and non-zero pressure, or non-zero gap and zero pressure
  $$g\,\sigma_n = 0$$
- **No shear transfer** *(automatically)*
  $$\underline{\boldsymbol{\sigma}}_t^{**} = 0$$



Scheme explaining normal contact conditions

$\sigma_n^* = (\underline{\underline{\boldsymbol{\sigma}}} \cdot \underline{\boldsymbol{n}}) \cdot \underline{\boldsymbol{n}} = \underline{\underline{\boldsymbol{\sigma}}} : (\underline{\boldsymbol{n}} \otimes \underline{\boldsymbol{n}})$

$\underline{\boldsymbol{\sigma}}_t^{**} = \underline{\underline{\boldsymbol{\sigma}}} \cdot \underline{\boldsymbol{n}} - \sigma_n \underline{\boldsymbol{n}} = \underline{\boldsymbol{n}} \cdot \underline{\underline{\boldsymbol{\sigma}}} \cdot \left(\underline{\underline{\boldsymbol{I}}} - \underline{\boldsymbol{n}} \otimes \underline{\boldsymbol{n}}\right)$

Frictionless or normal contact conditions

- **No penetration**
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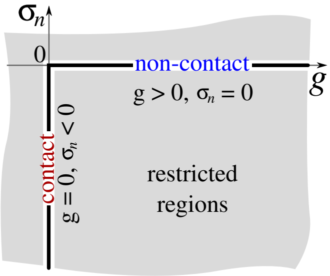

Improved scheme explaining normal contact conditions

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Frictionless or normal contact conditions

In **mechanics:**

Normal contact conditions

$\equiv$

Frictionless contact conditions

$\equiv$

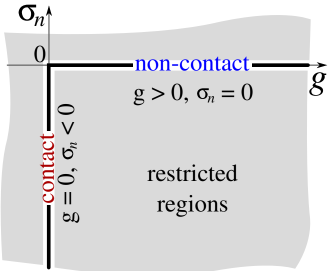
Hertz[1]*-Signorini*[2] *conditions*

$\equiv$

Hertz[1]*-Signorini*[2]*-Moreau*[3] *conditions*

also known in **optimization theory** as

Karush[4]*-Kuhn*[5]*-Tucker*[6] *conditions*



Improved scheme explaining normal contact conditions

$$\boxed{g \geq 0, \qquad \sigma_n \leq 0, \qquad g\sigma_n = 0}$$

[1] **Heinrich Rudolf Hertz** (1857–1894) a German physicist who first formulated and solved the frictionless contact problem between elastic ellipsoidal bodies.

[2] **Antonio Signorini** (1888–1963) an Italian mathematical physicist who gave a general and rigorous mathematical formulation of contact constraints.

[3] **Jean Jacques Moreau** (1923) a French mathematician who formulated a non-convex optimization problem based on these conditions and introduced pseudo-potentials in contact mechanics.

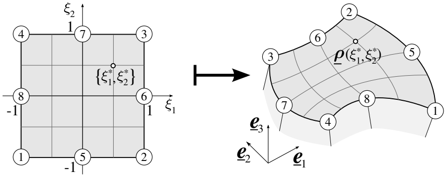
[4] **William Karush** (1917–1997), [5] **Harold William Kuhn** (1925) American mathematicians,

[6] **Albert William Tucker** (1905–1995) a Canadian mathematician.

Recall:

- Convective coordinate in parent space $\xi_i \in (-1;1)$
- Mapping to real space

$$\underline{\rho}(\xi_1,\xi_2,t) = \sum_{i=1}^{8} N^i(\xi_1,\xi_2)\underline{\rho}^i(t)$$



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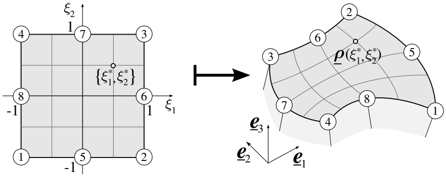
- Mapping to real space

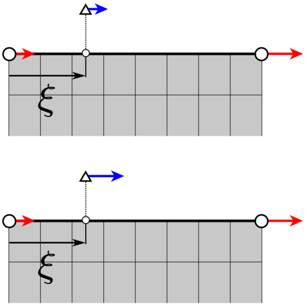
$$\underline{\rho}(\xi_1,\xi_2,t) = \sum_{i=1}^{8} N^i(\xi_1,\xi_2)\underline{\rho}^i(t)$$



- **Tangential slip velocity $\underline{v}_t$** must take into account:
  - only tangential component
  - relative rigid body motion
  - master's deformation

$$\underline{v}_t = \frac{\partial \underline{\rho}}{\partial \xi_1}\dot{\xi}_1 + \frac{\partial \underline{\rho}}{\partial \xi_2}\dot{\xi}_2$$

where $\partial\underline{\rho}/\partial\xi_i$ are the tangent vectors of the local basis and ξ_i are the convective coordinates.



Relative slip between a slave point and a deformable master surface

Relative sliding: example

Consider a one-dimensional example:
P is a projection of A on segment BC.

$$x_P = \xi x_C + (1-\xi)x_B \qquad (1)$$

Velocity of the projection point

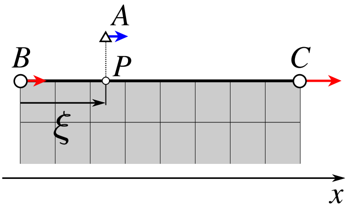
$$\dot x_P = \underbrace{\xi \dot x_C + (1-\xi)\dot x_B}_{\frac{\partial x_P}{\partial t}} + \underbrace{(x_C - x_B)\dot\xi}_{\frac{\partial x_P}{\partial \xi}\dot\xi}$$

Substract the velocity of point x_P for fixed ξ

$$v_t = \dot x_P - \frac{\partial x_P}{\partial t} = (x_C - x_B)\dot\xi = \frac{\partial x}{\partial \xi}\dot\xi$$

Compute tangential slip increment

$$\Delta g_t^{n+1} = \left.\frac{\partial x}{\partial \xi}\right|_{\xi^n}(\xi^{n+1} - \xi^n)$$



Example of a one-dimensional relative slip

Relative sliding: example

Consider a one-dimensional example:
P is a projection of A on segment BC.

$$x_P = \xi x_C + (1-\xi)x_B \qquad (1)$$

Velocity of the projection point

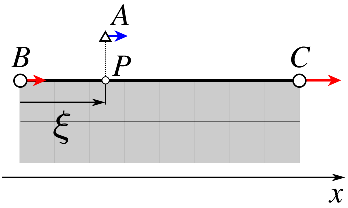
$$\dot x_P = \underbrace{\xi \dot x_C + (1-\xi)\dot x_B}_{\frac{\partial x_P}{\partial t}} + \underbrace{(x_C - x_B)\dot\xi}_{\frac{\partial x_P}{\partial \xi}\dot\xi}$$

Substract the velocity of point x_P for fixed ξ

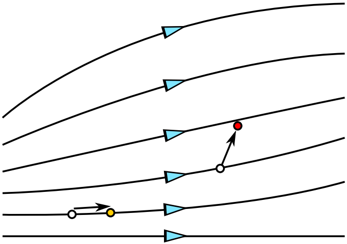
$$v_t = \dot x_P - \frac{\partial x_P}{\partial t} = (x_C - x_B)\dot\xi = \frac{\partial x}{\partial \xi}\dot\xi$$

Compute tangential slip increment

$$\Delta g_t^{n+1} = \left.\frac{\partial x}{\partial \xi}\right|_{\xi^n}(\xi^{n+1} - \xi^n)$$



Example of a one-dimensional relative slip



Fisherman's analogy: observing sea flow around the boat.
Lie derivative: the change of a vector field along the change of another vector field

- **No contact** $g > 0,\ \sigma_n = 0$
- **Stick** $|\underline{v}_t| = 0$

  Inside slip surface/Coulomb's cone

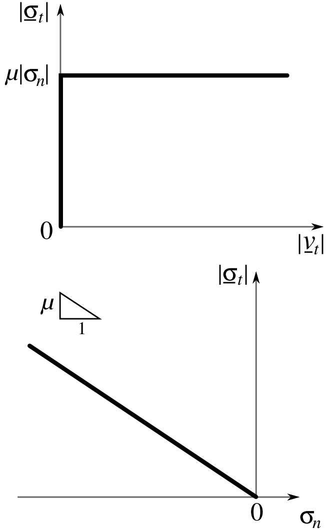
  $$f = |\underline{\sigma}_t| - \mu|\sigma_n| < 0$$
- **Slip** $|\underline{v}_t| > 0$

  On slip surface/Coulomb's cone

  $$f = |\underline{\sigma}_t| - \mu|\sigma_n| = 0$$
- **Complementary condition**

  Either zero velocity and negative slip criterion, or non-zero velocity and zero slip criterion

  $$|\underline{v}_t|\Big(|\underline{\sigma}_t| - \mu|\sigma_n|\Big) = 0$$



Scheme explaining frictional contact conditions

- **No contact** $g > 0,\ \sigma_n = 0$
- **Stick** $|\underline{v}_t| = 0$

  Inside slip surface/Coulomb's cone

  $$f = |\underline{\sigma}_t| - \mu|\sigma_n| < 0$$

- **Slip** $|\underline{v}_t| > 0$

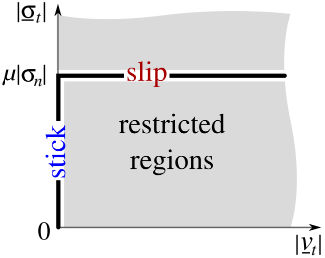
  On slip surface/Coulomb's cone

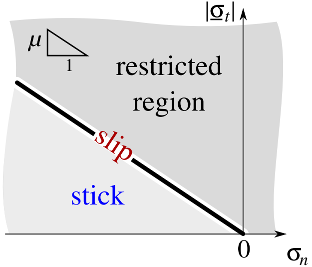
  $$f = |\underline{\sigma}_t| - \mu|\sigma_n| = 0$$

- **Complementary condition**

  Either zero velocity and negative slip criterion, or non-zero velocity and zero slip criterion

  $$|\underline{v}_t|\Big(|\underline{\sigma}_t| - \mu|\sigma_n|\Big) = 0$$





Improved scheme explaining frictional contact conditions

- **No contact** $g > 0,\ \sigma_n = 0$
- **Stick** $|\underline{v}_t| = 0$

  Inside slip surface/Coulomb's cone

  $$f = |\underline{\sigma}_t| - \mu|\sigma_n| < 0$$

- **Slip** $|\underline{v}_t| > 0$

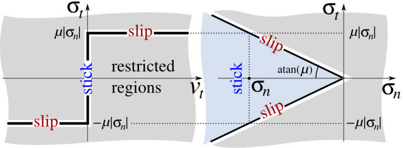
  On slip surface/Coulomb's cone

  $$f = |\underline{\sigma}_t| - \mu|\sigma_n| = 0$$

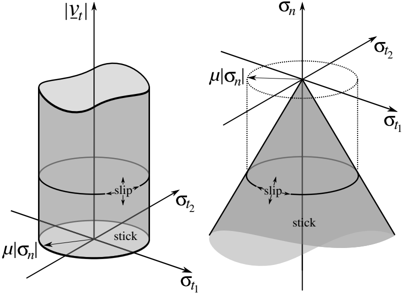
- **Complementary condition**

  Either zero velocity and negative slip criterion, or non-zero velocity and zero slip criterion

  $$|\underline{v}_t|\Big(|\underline{\sigma}_t| - \mu|\sigma_n|\Big) = 0$$



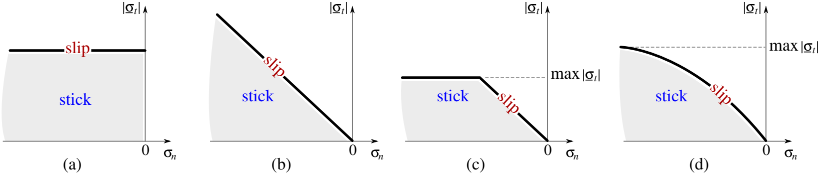
Scheme of 2D frictional contact



Scheme of 3D frictional contact

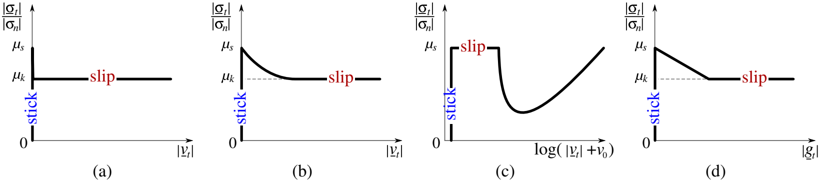
$$|\underline{v}_t| \geq 0, \quad |\underline{\sigma}_t| - \mu|\sigma_n| \leq 0, \quad |\underline{v}_t|\Big(|\underline{\sigma}_t| - \mu|\sigma_n|\Big) = 0 \quad \frac{\underline{\sigma}_t}{|\underline{\sigma}_t|} = -\frac{\underline{v}_t}{|\underline{v}_t|}$$

- **Static criteria**



(a) Tresca (b) Amontons-Coulomb (c) Coulomb-Orowan (d) Shaw

- **Kinetic criteria**



(a,b) velocity weakening (c) velocity weakening-strengthening
(d) Linear slip weakening

- μ_s static and μ_k kinetic coefficients of friction.

Rate and state friction and regularization

- **Rate and state friction law**
  - Rate $v_t = |\underline{v}_t|$ – relative slip velocity
  - State θ – ≈ internal time
  - Dieterich–Ruina–Perrin (1979, 83, 95)

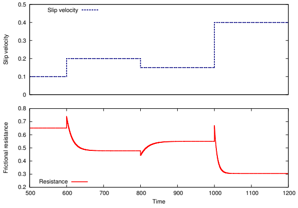
    Frictional resistance

    $$\sigma_t^c = |\sigma_n| \left[\mu_s + b\theta + a \ln(v_t/v_0)\right]$$

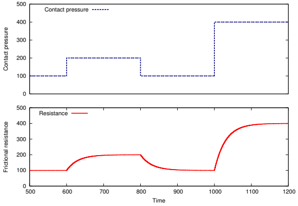
    Evolution of the state variable

    $$\dot\theta = -\frac{v_t}{L}\left[\theta + \ln\left(\frac{v_t}{v_0}\right)\right]$$

- **Prakash-Clifton friction law (1992,2000)**
  - Viscous type evolution of frictional resistance σ_t
  - $\dot\sigma_t = -\frac{v_t}{L}(\sigma_t + \mu\sigma_n)$



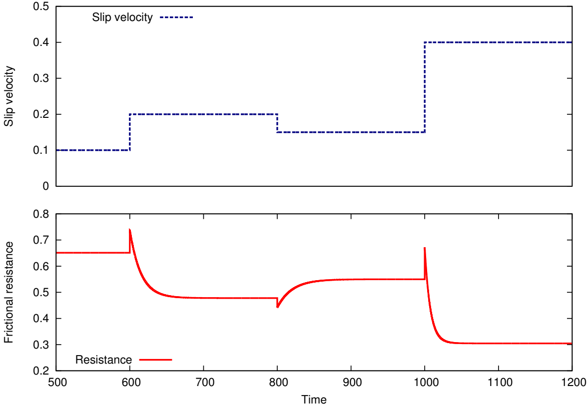
Rate and state friction law



Prakash-Clifton regularization

Rate and state friction and regularization

- **Rate and state friction law**

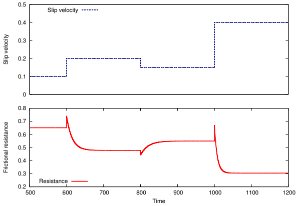


Rate and state friction and regularization

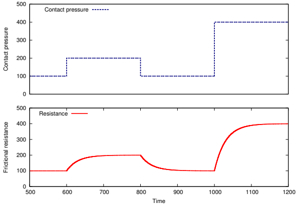
- **Rate and state friction law**
  - Rate $v_t = |\underline{v}_t|$ – relative slip velocity
  - State θ – $\approx$ internal time
  - Dieterich–Ruina–Perrin (1979, 83, 95)

    Frictional resistance
    $$\sigma_t^c = |\sigma_n| \left[\mu_s + b\theta + a\ln(v_t/v_0)\right]$$
    Evolution of the state variable
    $$\dot\theta = -\frac{v_t}{L}\left[\theta + \ln\left(\frac{v_t}{v_0}\right)\right]$$

- **Prakash-Clifton friction law (1992,2000)**
  - Viscous type evolution of frictional resistance σ_t
  - $\dot\sigma_t = -\frac{v_t}{L}(\sigma_t + \mu\sigma_n)$

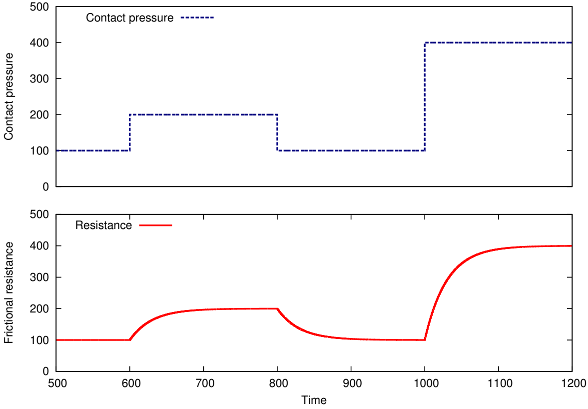


Rate and state friction law



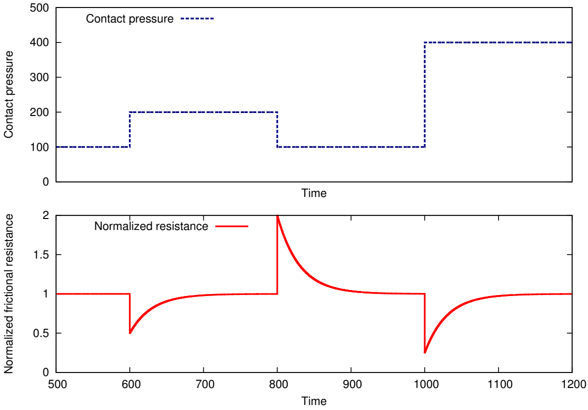
Prakash-Clifton regularization

- Prakash-Clifton friction law (1992,2000)

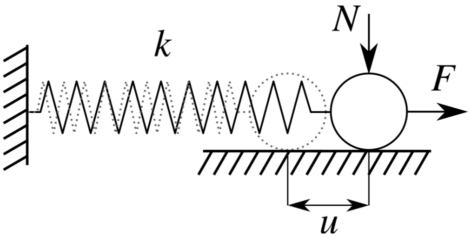


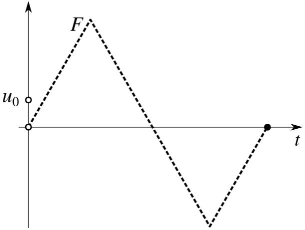
Rate and state friction and regularization

- Prakash-Clifton friction law (1992,2000)



Problem: find the point trajectory for a given force evolution



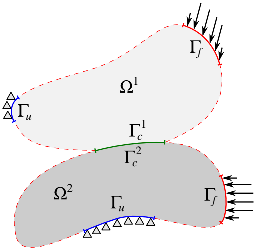


Towards a weak form

From strong to a weak form

- Balance of momentum and boundary conditions

  $\nabla \cdot \underline{\underline{\sigma}} + \underline{f}_v = 0 \quad \text{in } \Omega = \Omega_1 \cup \Omega_2 \qquad + \text{B.C.}$



Two solids in contact

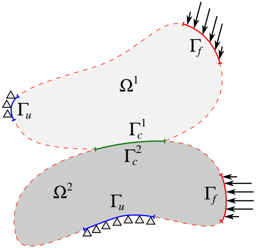
From strong to a weak form

- Balance of momentum and boundary conditions

$$\nabla \cdot \underline{\underline{\sigma}} + \underline{f}_v = 0 \quad \text{in } \Omega = \Omega_1 \cup \Omega_2 \qquad + \text{B.C.}$$

- Balance of virtual works 

$$\boxed{\int\limits_{\partial\Omega} \underline{n} \cdot \underline{\underline{\sigma}} \cdot \delta \underline{u} \, d\Gamma} + \int\limits_{\Omega} \left[\underline{f}_v \cdot \delta \underline{u} - \underline{\underline{\sigma}} \cdot\cdot \, \delta \nabla \underline{u} \right] d\Omega = 0$$



Two solids in contact

From strong to a weak form

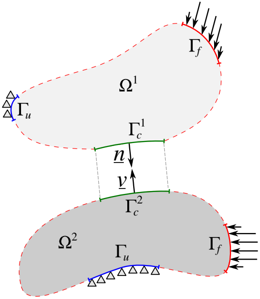
- Balance of momentum and boundary conditions

$$\nabla \cdot \underline{\underline{\sigma}} + \underline{f}_v = 0 \quad \text{in } \Omega = \Omega_1 \cup \Omega_2 \qquad + \text{B.C.}$$

- Balance of virtual works

$$\boxed{\int\limits_{\partial\Omega} \underline{n} \cdot \underline{\underline{\sigma}} \cdot \delta \underline{u} \, d\Gamma} =$$

$$\boxed{\int\limits_{\overline{\Gamma}_c^1} \underline{n} \cdot \underline{\underline{\sigma}} \cdot \delta \underline{\rho} \, d\overline{\Gamma}_c^1 + \int\limits_{\overline{\Gamma}_c^2} \underline{\nu} \cdot \underline{\underline{\sigma}} \cdot \delta \underline{r} \, d\overline{\Gamma}_c^2} + \int\limits_{\Gamma_f} \underline{\sigma}_0 \cdot \delta \underline{u} \, d\Gamma_f$$



Two solids in contact

From strong to a weak form

- Balance of momentum and boundary conditions

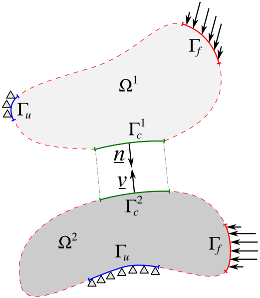
$$\nabla \cdot \underline{\underline{\sigma}} + \underline{f}_v = 0 \quad \text{in } \Omega = \Omega_1 \cup \Omega_2 \qquad + \text{B.C.}$$

- Balance of virtual works 

$$\boxed{\int\limits_{\partial\Omega} \underline{n} \cdot \underline{\underline{\sigma}} \cdot \delta\underline{u}\, d\Gamma} \Rightarrow$$

$$\boxed{\int\limits_{\overline{\Gamma}_c^1} \underline{n} \cdot \underline{\underline{\sigma}} \cdot \delta\underline{\rho}\, d\overline{\Gamma}_c^1 + \int\limits_{\overline{\Gamma}_c^2} \underline{\nu} \cdot \underline{\underline{\sigma}} \cdot \delta\underline{r}\, d\overline{\Gamma}_c^2} =$$

$$= \int\limits_{\overline{\Gamma}_c^1} \underline{n} \cdot \underline{\underline{\sigma}} \cdot \delta(\underline{\rho} - \underline{r})\, d\overline{\Gamma}_c^1 = \int\limits_{\overline{\Gamma}_c^1} \left(\sigma_n \delta g_n + \underset{\sim}{\sigma}{}_t^T \delta \underset{\sim}{\xi} \right) d\overline{\Gamma}_c^1$$



Two solids in contact

From strong to a weak form

- Balance of momentum and boundary conditions

$$\nabla \cdot \underline{\underline{\sigma}} + \underline{f}_v = 0 \quad \text{in } \Omega = \Omega_1 \cup \Omega_2 \qquad + \text{B.C.}$$

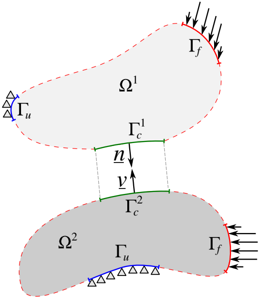
- Balance of virtual works 

$$\boxed{\int\limits_{\partial\Omega} \underline{n} \cdot \underline{\underline{\sigma}} \cdot \delta\underline{u}\, d\Gamma} \Rightarrow$$

$$\boxed{\int\limits_{\overline{\Gamma}_c^1} \underline{n} \cdot \underline{\underline{\sigma}} \cdot \delta\underline{\rho}\, d\overline{\Gamma}_c^1 + \int\limits_{\overline{\Gamma}_c^2} \underline{\nu} \cdot \underline{\underline{\sigma}} \cdot \delta\underline{r}\, d\overline{\Gamma}_c^2} =$$

$$= \int\limits_{\overline{\Gamma}_c^1} \underline{n} \cdot \underline{\underline{\sigma}} \cdot \delta(\underline{\rho} - \underline{r})\, d\overline{\Gamma}_c^1 = \int\limits_{\overline{\Gamma}_c^1} \left(\sigma_n \delta g_n + \underset{\sim}{\sigma}_t^T \delta \underset{\sim}{\xi}\right) d\overline{\Gamma}_c^1$$

$$\int\limits_{\Omega} \underline{\underline{\sigma}} \cdot\cdot\, \delta\nabla\underline{u}\, d\Omega + \underbrace{\boxed{\int\limits_{\overline{\Gamma}_c^1} \left(\sigma_n \delta g_n + \underset{\sim}{\sigma}_t^T \delta \underset{\sim}{\xi}\right) d\overline{\Gamma}_c^1}}_{\text{Contact term}} = \int\limits_{\Gamma_f} \underline{\sigma}_0 \cdot \delta\underline{u}\, d\Gamma + \int\limits_{\Omega} \underline{f}_v \cdot \delta\underline{u}\, d\Omega$$



Two solids in contact

From strong to a weak form

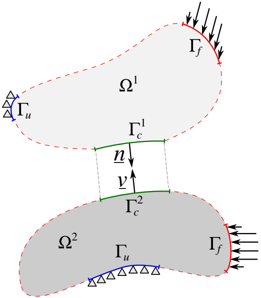
- Balance of momentum and boundary conditions

$$\nabla \cdot \underline{\underline{\sigma}} + \underline{f}_v = 0 \quad \text{in } \Omega = \Omega_1 \cup \Omega_2 \qquad + \text{B.C.}$$

- Balance of virtual works 

$$\underbrace{\int_{\Omega} \underline{\underline{\sigma}} \cdot\cdot \delta \nabla \underline{u} \, d\Omega}_{\text{Change of the internal energy}} + \underbrace{\boxed{\int_{\bar{\Gamma}_c^1} \left(\sigma_n \delta g_n + \underset{\sim}{\sigma}{}_t^T \delta \underset{\sim}{\xi} \right) d\bar{\Gamma}_c^1}}_{\text{Contact term}} =$$

$$\underbrace{\int_{\Gamma_f} \underline{\sigma}_0 \cdot \delta \underline{u} \, d\Gamma}_{\text{Virtual work of external forces}} + \underbrace{\int_{\Omega} \underline{f}_v \cdot \delta \underline{u} \, d\Omega}_{\text{Virtual work of volume forces}}$$



Two solids in contact

- Functional space

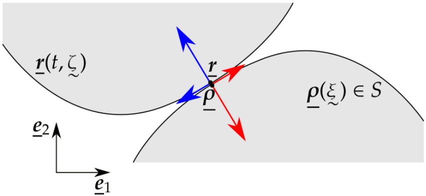
$\delta \underline{u} \in \mathbb{H}^1(\Omega)$ Hilbert space of the first order
and $\delta \underline{u}$ satisfy boundary and **contact conditions**.

Towards variational inequality

- Contact term

$$\int\limits_{\bar\Gamma_c^1} \left(\sigma_n \delta g_n + \underset{\sim}{\sigma}_t^T \delta \underset{\sim}{\xi}\right) d\bar\Gamma_c^1$$

$$\int\limits_{\bar\Gamma_c^1} \sigma_n \delta g_n \, d\bar\Gamma_c^1 = 0$$



Contact configuration $\sigma_n \delta g_n = 0, \; \sigma_n \leq 0$

$$\int\limits_{\Omega} \underline{\underline{\sigma}} \cdot\cdot \delta \nabla \underline{u} \, d\Omega + \int\limits_{\bar\Gamma_c^1} \underset{\sim}{\sigma}_t^T \delta \underset{\sim}{\xi} \, d\bar\Gamma_c^1 \geq \int\limits_{\Gamma_f} \underline{\sigma}_0 \cdot \delta \underline{u} \, d\Gamma + \int\limits_{\Omega} \underline{f}_v \cdot \delta \underline{u} \, d\Omega,$$

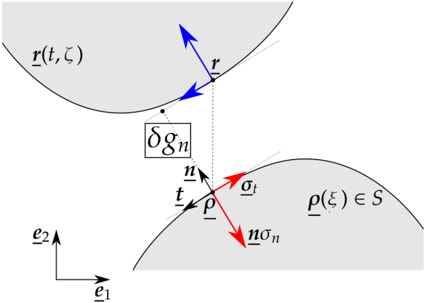
$$\mathbb{K} = \left\{ \delta \underline{u} \in \mathbb{H}^1(\Omega) \;\middle|\; \delta \underline{u} = 0 \text{ on } \Gamma_u, \; g_n(\underline{u} + \delta \underline{u}) \geq 0 \text{ on } \Gamma_c \right\}$$

Towards variational inequality

- Contact term

$$\int\limits_{\bar\Gamma_c^1} \left(\sigma_n \delta g_n + \underset{\sim}{\sigma}_t^T \delta \underset{\sim}{\xi}\right) d\bar\Gamma_c^1$$

$$\int\limits_{\bar\Gamma_c^1} \sigma_n \delta g_n \, d\bar\Gamma_c^1 = 0$$



Virtual change of the configuration

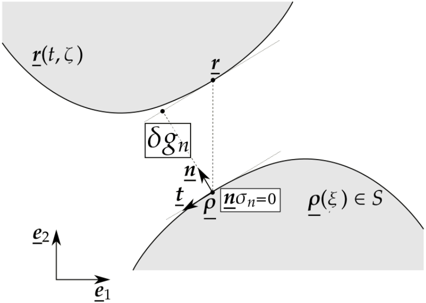
$$\int\limits_{\Omega} \underline{\underline{\sigma}} \cdot\cdot \delta\nabla\underline{u}\, d\Omega + \int\limits_{\bar\Gamma_c^1} \underset{\sim}{\sigma}_t^T \delta\underset{\sim}{\xi}\, d\bar\Gamma_c^1 \geq \int\limits_{\Gamma_f} \underline{\sigma}_0 \cdot \delta\underline{u}\, d\Gamma + \int\limits_{\Omega} \underline{f}_v \cdot \delta\underline{u}\, d\Omega,$$

$$\mathbb{K} = \left\{ \delta\underline{u} \in \mathbb{H}^1(\Omega) \;\middle|\; \delta\underline{u} = 0 \text{ on } \Gamma_u,\; g_n(\underline{u} + \delta\underline{u}) \geq 0 \text{ on } \Gamma_c \right\}$$

- Contact term

$$\int\limits_{\bar\Gamma_c^1} \left(\sigma_n \delta g_n + \underset{\sim}{\sigma}_t^T \delta \underset{\sim}{\xi}\right) d\bar\Gamma_c^1$$

$$\int\limits_{\bar\Gamma_c^1} \sigma_n \delta g_n \, d\bar\Gamma_c^1 = 0$$



Normal term in separation $\delta g_n > 0$

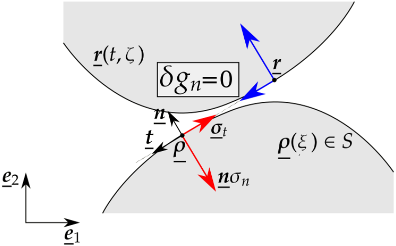
$$\int\limits_{\Omega} \underline{\underline{\sigma}} \cdot\cdot \delta\nabla\underline{u} \, d\Omega + \int\limits_{\bar\Gamma_c^1} \underset{\sim}{\sigma}_t^T \delta \underset{\sim}{\xi} \, d\bar\Gamma_c^1 \geq \int\limits_{\Gamma_f} \underline{\sigma}_0 \cdot \delta\underline{u} \, d\Gamma + \int\limits_{\Omega} \underline{f}_v \cdot \delta\underline{u} \, d\Omega,$$

$$\mathbb{K} = \left\{\delta\underline{u} \in \mathbb{H}^1(\Omega) \;\middle|\; \delta\underline{u} = 0 \text{ on } \Gamma_u,\; g_n(\underline{u} + \delta\underline{u}) \geq 0 \text{ on } \Gamma_c\right\}$$

- Contact term

$$\int\limits_{\bar\Gamma_c^1} \left(\sigma_n \delta g_n + \underset{\sim}{\sigma}_t^T \delta \underset{\sim}{\xi}\right) d\bar\Gamma_c^1$$

$$\int\limits_{\bar\Gamma_c^1} \sigma_n \delta g_n \, d\bar\Gamma_c^1 = 0$$



Normal term in sliding $\delta g_n = 0$

$$\int\limits_{\Omega} \underline{\underline{\sigma}} \cdot\cdot \delta\nabla\underline{u} \, d\Omega + \int\limits_{\bar\Gamma_c^1} \underset{\sim}{\sigma}_t^T \delta \underset{\sim}{\xi} \, d\bar\Gamma_c^1 \geq \int\limits_{\Gamma_f} \underline{\sigma}_0 \cdot \delta\underline{u} \, d\Gamma + \int\limits_{\Omega} \underline{f}_v \cdot \delta\underline{u} \, d\Omega,$$

$$\mathbb{K} = \left\{ \delta\underline{u} \in \mathbb{H}^1(\Omega) \;\middle|\; \delta\underline{u} = 0 \text{ on } \Gamma_u,\; g_n(\underline{u} + \delta\underline{u}) \geq 0 \text{ on } \Gamma_c \right\}$$

Back to variational equality (unconstrained)

- Constrained minimization problem

$$\int\limits_{\Omega} \underline{\underline{\sigma}} \cdot\cdot \delta\nabla\underline{u}\, d\Omega + \int\limits_{\bar{\Gamma}_c^1} \underset{\sim}{\sigma}_t^T \delta\underset{\sim}{\xi}\, d\bar{\Gamma}_c^1 \geq \int\limits_{\Gamma_f} \underline{\sigma}_0 \cdot \delta\underline{u}\, d\Gamma + \int\limits_{\Omega} \underline{f}_v \cdot \delta\underline{u}\, d\Omega,$$

$$\mathbb{K} = \left\{ \delta\underline{u} \in \mathbb{H}^1(\Omega) \;\middle|\; \delta\underline{u} = 0 \text{ on } \Gamma_u,\; g_n(\underline{u} + \delta\underline{u}) \geq 0 \text{ on } \Gamma_c \right\}$$

- Use optimization theory to convert to

$$\int\limits_{\Omega} \underline{\underline{\sigma}} \cdot\cdot \delta\nabla\underline{u}\, d\Omega + \int\limits_{\Gamma_c^1} \underbrace{\boxed{C(\sigma_n, \sigma_t, g_n, \underset{\sim}{\xi}, \delta\underline{u})}}_{\text{Contact term}^*} d\Gamma_c^1 = \int\limits_{\Gamma_f} \underline{\sigma}_0 \cdot \delta\underline{u}\, d\Gamma + \int\limits_{\Omega} \underline{f}_v \cdot \delta\underline{u}\, d\Omega,$$

Unconstrained functional space $\mathbb{K} = \left\{ \delta\underline{u} \in \mathbb{H}^1(\Omega) \;\middle|\; \delta\underline{u} = 0 \text{ on } \Gamma_u \right\}$

Contact term* is defined on the *potential contact zone* Γ_c^1.

Optimization methods

Optimization methods: recall

Functional to be minimized $F(\mathbf{x})$ under constraint $g(\mathbf{x}) \geq 0$

- Penalty method
- Lagrange multipliers method
- Augmented Lagrangian method

Optimization methods: recall

Functional to be minimized $F(\mathbf{x})$ under constraint $g(\mathbf{x}) \geq 0$

- **Penalty method**
  - New functional

$$F_p(\mathbf{x}) = F(\mathbf{x}) + \boxed{\epsilon \langle -g(\mathbf{x}) \rangle^2} = F(\mathbf{x}) + \begin{cases} 0, & \text{if } g(\mathbf{x}) \geq 0 \quad \textit{non-contact} \\ \epsilon g^2(\mathbf{x}), & \text{if } g(\mathbf{x}) < 0 \quad \textit{contact} \end{cases}$$

  where ϵ is the penalty parameter.
  - Stationary point must satisfy

$$\nabla F_p(\mathbf{x}) = \nabla F(\mathbf{x}) + 2\epsilon \langle -g(\mathbf{x}) \rangle \nabla g(\mathbf{x}) = 0$$

  - Solution **tends** to the precise solution as $\epsilon \to \infty$

- Lagrange multipliers method
- Augmented Lagrangian method

Macaulay brackets $\langle x \rangle = \begin{cases} x, & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases}$

Functional to be minimized $F(\mathbf{x})$ under constraint $g(\mathbf{x}) \ge 0$

- Penalty method $F_p(\mathbf{x}) = F(\mathbf{x}) + \boxed{\epsilon \langle -g(\mathbf{x})\rangle^2}$
- **Lagrange multipliers method**
  - New functional called **Lagrangian**

$$\mathcal{L}(\mathbf{x}, \lambda) = F(\mathbf{x}) + \boxed{\lambda g(\mathbf{x})}$$

  - Saddle point problem

$$\min_{x} \max_{\lambda} \{\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda})\} \longrightarrow \mathbf{x}^* \longleftarrow \min_{g(\mathbf{x})\ge 0} \{F(\mathbf{x})\}$$

  - Stationary point

$$\nabla_{\mathbf{x},\lambda} \mathcal{L} = \begin{bmatrix} \nabla_{\mathbf{x}} F(\mathbf{x}) + \lambda \nabla_x g(\mathbf{x}) \\ g(\mathbf{x}) \end{bmatrix} = 0 \quad \text{need to verify} \quad \lambda \le 0$$

- Augmented Lagrangian method

Macaulay brackets $\langle x \rangle = \begin{cases} x, & \text{if } x \ge 0 \\ 0, & \text{otherwise} \end{cases}$

Optimization methods: recall

Functional to be minimized $F(\mathbf{x})$ under constraint $g(\mathbf{x}) \geq 0$

- Penalty method $F_p(\mathbf{x}) = F(\mathbf{x}) + \boxed{\epsilon \left\langle -g(\mathbf{x}) \right\rangle^2}$
- Lagrange multipliers method $\mathcal{L}(\mathbf{x}, \lambda) = F(\mathbf{x}) + \boxed{\lambda g(\mathbf{x})}$
- **Augmented Lagrangian method**
  [Hestnes 1969], [Powell 1969], [Glowinski & Le Tallec 1989], [Alart & Curnier 1991], [Simo & Laursen 1992]
  - New functional, augmented Lagrangian

$$\mathcal{L}_a(\mathbf{x}, \lambda) = F(\mathbf{x}) + \begin{cases} \boxed{\lambda g(\mathbf{x})} + \boxed{\epsilon g^2(\mathbf{x})}, & \text{if } \lambda + 2\epsilon g(\mathbf{x}) \geq 0, \ \textit{contact} \\ -\frac{1}{4\epsilon}\lambda^2, & \text{if } \lambda + 2\epsilon g(\mathbf{x}) < 0, \ \textit{non-contact} \end{cases}$$

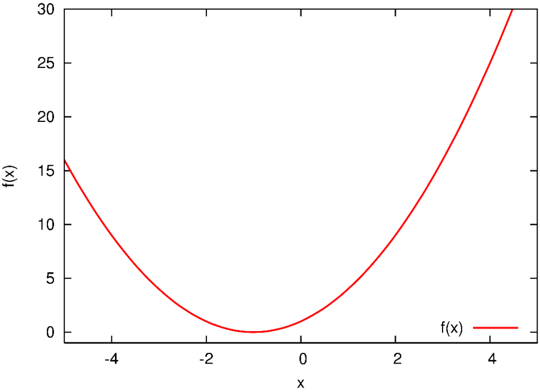
  - Stationary point

$$\nabla_{\mathbf{x},\lambda} \mathcal{L}_a = \begin{cases} \begin{bmatrix} \nabla_{\mathbf{x}} F(\mathbf{x}) + \lambda \nabla_x g(\mathbf{x}) + 2\epsilon g(\mathbf{x}) \nabla g(\mathbf{x}) \\ g(\mathbf{x}) \end{bmatrix} = 0, & \textit{if contact} \\ \begin{bmatrix} \nabla_{\mathbf{x}} F(\mathbf{x}) \\ -\frac{\lambda}{\epsilon} \end{bmatrix} = 0, & \textit{if non-contact} \end{cases}$$

Macaulay brackets $\langle x \rangle = \begin{cases} x, & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases}$

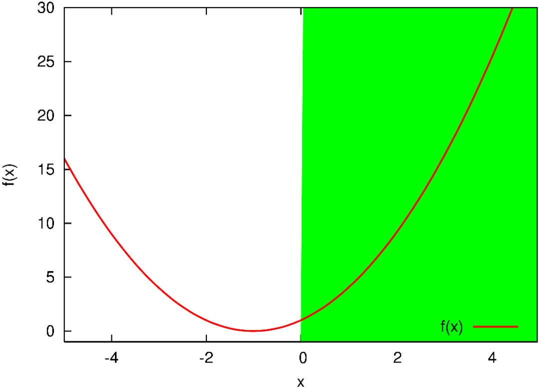
Uzawa algorithm

Optimization methods: example



$$\textbf{Functional :}\ f(x) = x^2 + 2x + 1$$
$$\text{Constrain : } g(x) = x \geq 0$$
$$\text{Solution : } x^* = 0$$

Optimization methods: example



Functional : $f(x) = x^2 + 2x + 1$

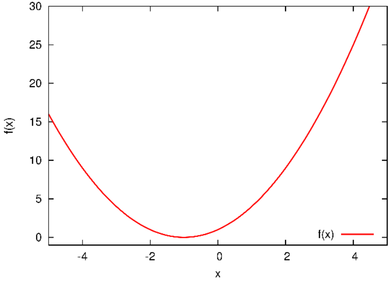
Constrain : $g(x) = x \geq 0$

Solution : $x^* = 0$

$$F(x) = x^2 + 2x + 1, \;\; g(x) = x \ge 0, \;\; x^* = 0$$

- **Penalty method**

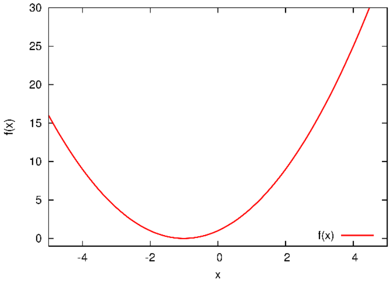
$$F_p(x) = F(x) + \boxed{\epsilon \left\langle -g(x) \right\rangle^2}$$

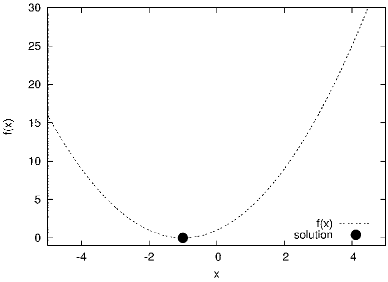


$$F(x) = x^2 + 2x + 1, \;\; g(x) = x \geq 0, \;\; x^* = 0$$

- **Penalty method**

$$F_p(x) = F(x) + \boxed{\epsilon \left\langle -g(x) \right\rangle^2}$$



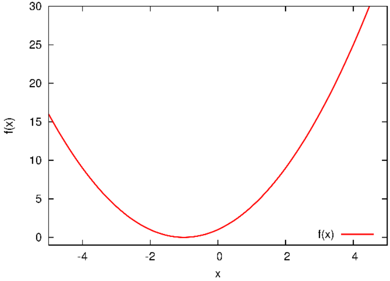


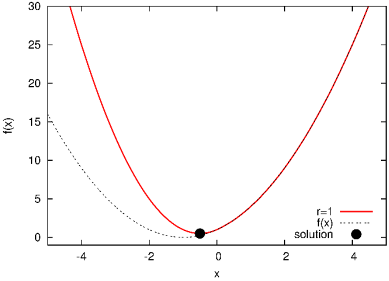
$$\epsilon = 0$$

$$F(x) = x^2 + 2x + 1, \;\; g(x) = x \geq 0, \;\; x^* = 0$$

- **Penalty method**

$$F_p(x) = F(x) + \boxed{\epsilon \langle -g(x) \rangle^2}$$





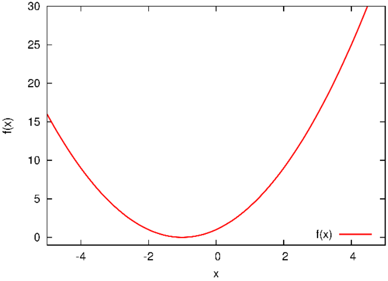
$\epsilon = 1$

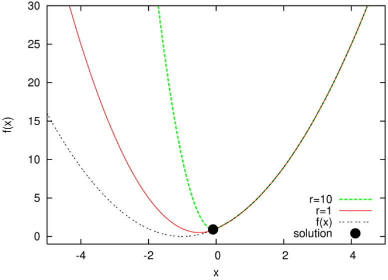
Penalty method: example

$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

- **Penalty method**

$$F_p(x) = F(x) + \boxed{\epsilon \left\langle -g(x) \right\rangle^2}$$





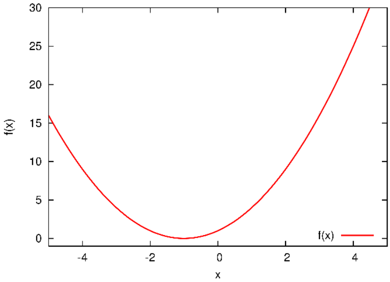
$\epsilon = 10$

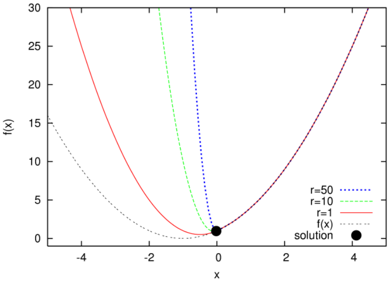
Penalty method: example

$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

- **Penalty method**

$$F_p(x) = F(x) + \boxed{\epsilon \langle -g(x) \rangle^2}$$





$\epsilon = 50$

Penalty method: example

$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

- **Penalty method**

$$F_p(x) = F(x) + \boxed{\epsilon \langle -g(x) \rangle^2}$$

Advantages ☺

- simple physical interpretation
- simple implementation
- no additional degrees of freedom
- "mathematically" smooth functional

Drawbacks ☹

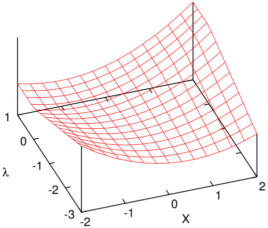
- practically non-smooth functional
- solution is not exact:
  - too small penalty → large penetration
  - too large penalty → ill-conditioning of the tangent matrix
- user has to choose penalty ϵ properly or automatically and/or adapt during convergence

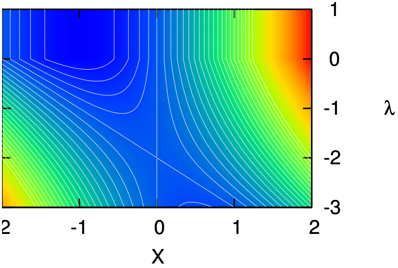
$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

- **Lagrange multipliers method**

  $\mathcal{L}(x,\lambda) = F(x) + \boxed{\lambda g(x)}$ $\rightarrow$ Saddle point $\rightarrow \min\limits_{x}\max\limits_{\lambda}\mathcal{L}(x,\lambda)$

  Need to check that $\lambda \leq 0$





Lagrange multipliers method: example

$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

- **Lagrange multipliers method**

  $\mathcal{L}(x,\lambda) = F(x) + \boxed{\lambda g(x)}$ $\rightarrow$ Saddle point $\rightarrow$ $\min\limits_{x}\max\limits_{\lambda}\mathcal{L}(x,\lambda)$

  Need to check that $\lambda \leq 0$

Advantages ☺

- exact solution
- no adjustable parameters

Drawbacks ☹

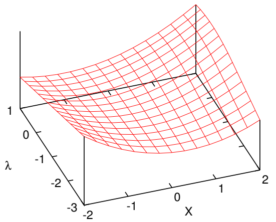
- Lagrangian is not smooth
- additional degrees of freedom
- not fully unconstrained: $\lambda \leq 0$

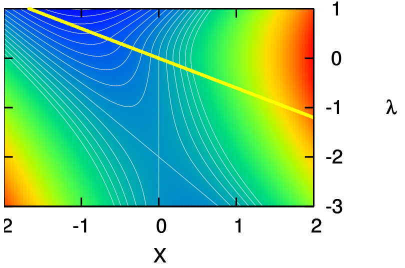
Augmented Lagrangian method: example

$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

- **Augmented Lagrangian method**

$$\mathcal{L}_a(\mathbf{x},\lambda) = F(\mathbf{x}) + \begin{cases} \boxed{\lambda g(\mathbf{x})} + \boxed{\epsilon g^2(\mathbf{x})}, & \text{if } \lambda + 2\epsilon g(\mathbf{x}) \geq 0, \textit{ contact} \\ -\frac{1}{4\epsilon}\lambda^2, & \text{if } \lambda + 2\epsilon g(\mathbf{x}) < 0, \textit{ non-contact} \end{cases}$$



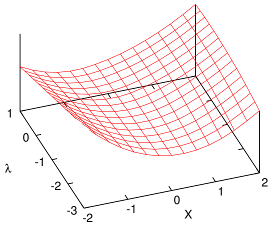


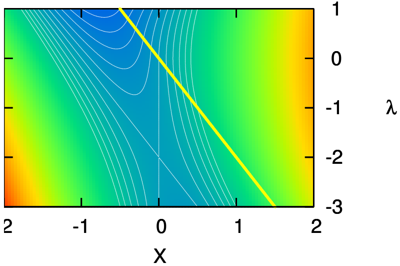
Yellow line separates contact and non-contact regions

$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

- **Augmented Lagrangian method**

$$\mathcal{L}_a(\mathbf{x}, \lambda) = F(\mathbf{x}) + \begin{cases} \boxed{\lambda g(\mathbf{x})} + \boxed{\epsilon g^2(\mathbf{x})}, & \text{if } \lambda + 2\epsilon g(\mathbf{x}) \geq 0, \textit{ contact} \\ -\frac{1}{4\epsilon}\lambda^2, & \text{if } \lambda + 2\epsilon g(\mathbf{x}) < 0, \textit{ non-contact} \end{cases}$$





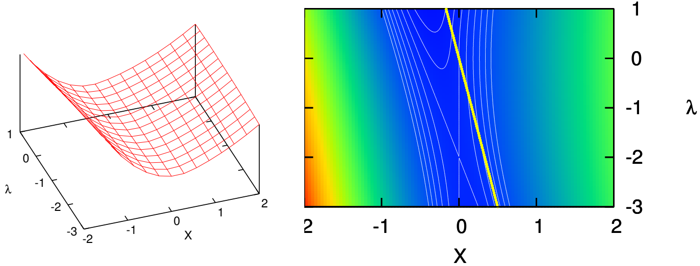
Yellow line separates contact and non-contact regions

Augmented Lagrangian method: example

$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

- **Augmented Lagrangian method**

$$\mathcal{L}_a(\mathbf{x}, \lambda) = F(\mathbf{x}) + \begin{cases} \boxed{\lambda g(\mathbf{x})} + \boxed{\epsilon g^2(\mathbf{x})}, & \text{if } \lambda + 2\epsilon g(\mathbf{x}) \geq 0, \textit{ contact} \\ -\frac{1}{4\epsilon}\lambda^2, & \text{if } \lambda + 2\epsilon g(\mathbf{x}) < 0, \textit{ non-contact} \end{cases}$$



Yellow line separates contact and non-contact regions

Augmented Lagrangian method: example

$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

- **Augmented Lagrangian method**

$$\mathcal{L}_a(\mathbf{x}, \lambda) = F(\mathbf{x}) + \begin{cases} \boxed{\lambda g(\mathbf{x})} + \boxed{\epsilon g^2(\mathbf{x})}, & \text{if } \lambda + 2\epsilon g(\mathbf{x}) \geq 0, \; \textit{contact} \\ -\frac{1}{4\epsilon}\lambda^2, & \text{if } \lambda + 2\epsilon g(\mathbf{x}) < 0, \; \textit{non-contact} \end{cases}$$

Advantages ☺

- exact solution
- smooth functional (!)
- fully unconstrained

Drawbacks ☹

- additional degrees of freedom
- quite sensitive to parameter ϵ
- need to adjust ϵ during convergence

Application to contact problems: weak form

$$\int\limits_{\Omega} \underline{\underline{\sigma}} \cdot\cdot \delta\nabla\underline{u}\, d\Omega + \int\limits_{\Gamma_c^1} \underbrace{\boxed{\mathrm{C}}}_{\text{Contact term}} d\Gamma_c^1 = \int\limits_{\Gamma_f} \underline{\sigma}_0 \cdot \delta\underline{u}\, d\Gamma + \int\limits_{\Omega} \underline{f}_v \cdot \delta\underline{u}\, d\Omega,$$

$$\mathbb{K} = \left\{\delta\underline{u} \in \mathbb{H}^1(\Omega) \;\middle|\; \delta\underline{u} = 0 \;\; \text{on } \Gamma_u\right\}$$

- Penalty method

Pressure: $\sigma_n = \epsilon g_n$, Shear: $\underline{\sigma}_t = \begin{cases} \epsilon \underline{g}_t, & \textit{if stick } |\sigma_t| < \mu|\sigma_n| \\ \mu\epsilon g_n \delta\underline{g}_t / |\delta\underline{g}_t|, & \textit{if slip } |\sigma_t| = \mu|\sigma_n| \end{cases}$

Contact term

$$C = C(g_n, \underline{g}_t, \delta g_n, \delta\underline{g}_t) = \sigma_n \delta g_n + \underline{\sigma}_t \cdot \delta\underline{g}_t$$

Application to contact problems: weak form

$$\int\limits_{\Omega} \underline{\underline{\sigma}} \cdot\cdot \delta \nabla \underline{u} \, d\Omega + \int\limits_{\Gamma_c^1} \underbrace{\boxed{C}}_{\text{Contact term}} d\Gamma_c^1 = \int\limits_{\Gamma_f} \underline{\sigma}_0 \cdot \delta \underline{u} \, d\Gamma + \int\limits_{\Omega} \underline{f}_v \cdot \delta \underline{u} \, d\Omega,$$

$$\mathbb{K} = \left\{ \delta \underline{u} \in \mathbb{H}^1(\Omega) \;\middle|\; \delta \underline{u} = 0 \;\text{ on } \Gamma_u \right\}$$

- Augmented Lagrangian method

Contact term

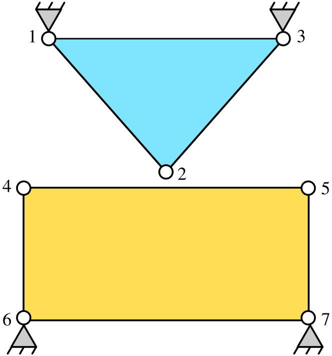
$C = C(g_n, \underline{g}_t, \lambda_n, \underline{\lambda}_t, \delta g_n, \delta \underline{g}_t, \delta \lambda_n, \delta \underline{\lambda}_t)$

$$C = \begin{cases} -\frac{1}{\epsilon}\left(\lambda_n \delta \lambda_n - \underline{\lambda}_t \cdot \delta \underline{\lambda}_t\right), & \textit{if non-contact } \lambda_n + \epsilon g_n \ge 0 \\ \hat{\lambda}_n \delta g_n + g_n \delta \lambda_n + \underline{\hat{\lambda}_t} \cdot \delta \underline{g}_t + \underline{g}_t \cdot \delta \underline{\hat{\lambda}_t}, & \textit{if stick } |\underline{\hat{\lambda}}_t| \le \mu |\hat{\sigma}_n| \\ \hat{\lambda}_n \delta g_n + g_n \delta \lambda_n + \mu \hat{\sigma}_n - \mu \hat{\sigma}_n \frac{\underline{\hat{\lambda}_t}}{|\underline{\hat{\lambda}_t}|} \cdot \delta \underline{g}_t - \frac{1}{\epsilon}\left(\lambda_t + \mu \hat{\sigma}_n \frac{\underline{\hat{\lambda}_t}}{|\underline{\hat{\lambda}_t}|}\right) \cdot \delta \underline{\lambda}_t, & \textit{if slip } |\underline{\hat{\lambda}}_t| \ge \mu |\hat{\sigma}_n| \end{cases}$$

where $\hat{\lambda}_n = \lambda_n + \epsilon g_n$ and $\underline{\hat{\lambda}}_t = \underline{\lambda}_t + \epsilon \underline{g}_t$.

Example

- Use penalty method to enforce Dirichlet BC
- Use penalty method to enforce contact constraints
- First, detect contact elements
- Second, construct updated residual vector and tangent matrix

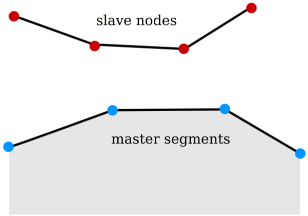


Contact between two elements

Detection

Introduction

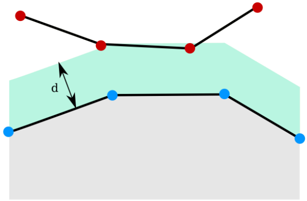
- Important and time consuming part
- With which master segment the slave node can/will come in contact?
- Need to know it in advance
- To reduce time:
  - Bounding boxes for the global search
  - Maximal distance of detection



Slave and master

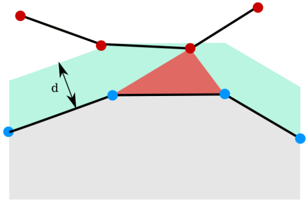
Introduction

- Important and time consuming part
- With which master segment the slave node can/will come in contact?
- Need to know it in advance
- To reduce time:
  - Bounding boxes for the global search
  - Maximal distance of detection



Slave in close zone

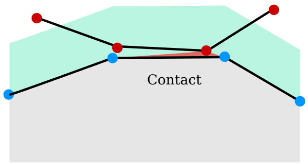
- Important and time consuming part
- With which master segment the slave node can/will come in contact?
- Need to know it in advance
- To reduce time:
  - Bounding boxes for the global search
  - Maximal distance of detection



NTS contact element

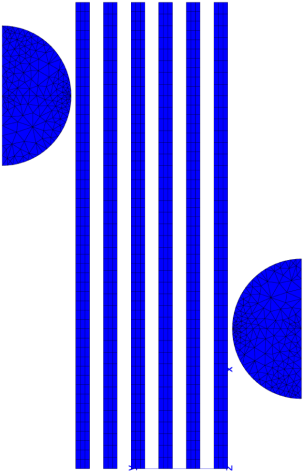
Introduction

- Important and time consuming part
- With which master segment the slave node can/will come in contact?
- Need to know it in advance
- To reduce time:
  - Bounding boxes for the global search
  - Maximal distance of detection



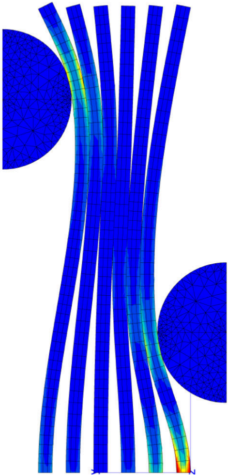
Contact occurs

- Important and time consuming part
- With which master segment the slave node can/will come in contact?
- Need to know it in advance
- To reduce time:
  - Bounding boxes for the global search
  - Maximal distance of detection



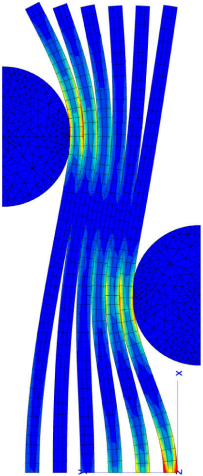
Multi-plate contact
Zset/Zébulon

- Important and time consuming part
- With which master segment the slave node can/will come in contact?
- Need to know it in advance
- To reduce time:
  - Bounding boxes for the global search
  - Maximal distance of detection



Multi-plate contact
Zset/Zébulon

- Important and time consuming part
- With which master segment the slave node can/will come in contact?
- Need to know it in advance
- To reduce time:
    - Bounding boxes for the global search
    - Maximal distance of detection



Multi-plate contact
Zset/Zébulon

- Important and time consuming part
- With which master segment the slave node can/will come in contact?
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- To reduce time:
    - Bounding boxes for the global search
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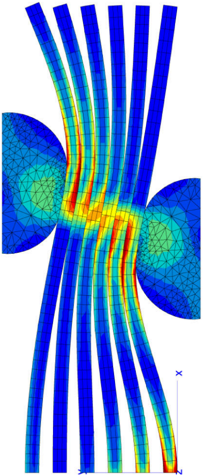

Multi-plate contact Zset/Zébulon

Introduction

- Important and time consuming part
- With which master segment the slave node can/will come in contact?
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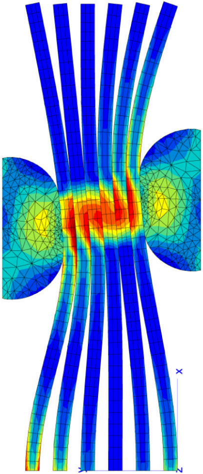

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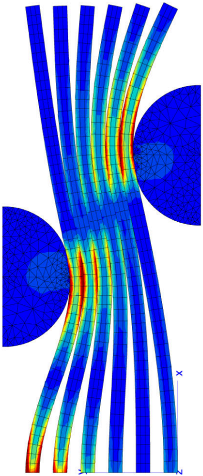

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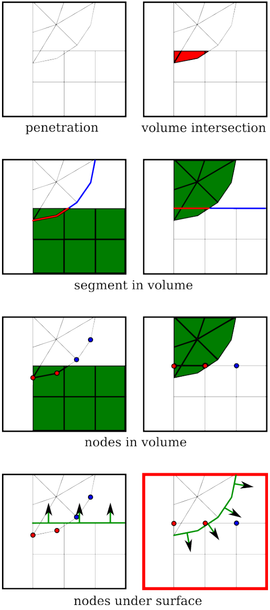
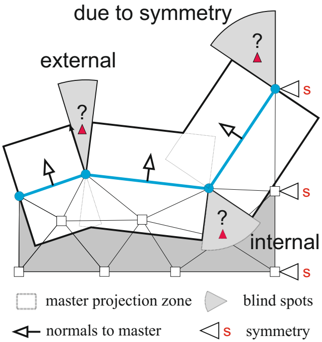
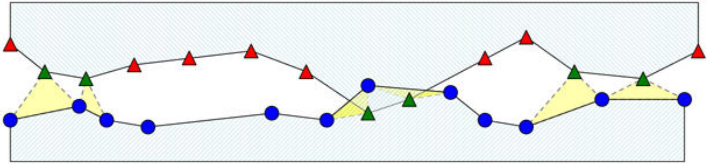


All-to-all approach

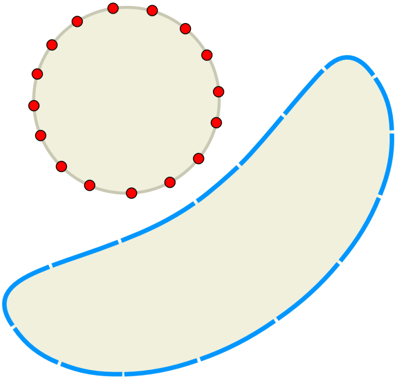
- Growth rate $O(N \times M)$
- Not robust
- Blind spots
- Slow





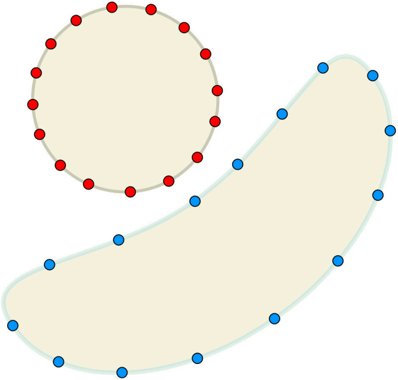
Closest node approach

- Node-to-segment detection ⇒ iterative solution



Closest node approach

- Node-to-segment detection ⇒ iterative solution
- Detection based on the closest node:
  1. find the closest master node;
  2. find a projection on segments adjacent to this node.
- Widely accepted simplification
- Simple treatment of blind spots



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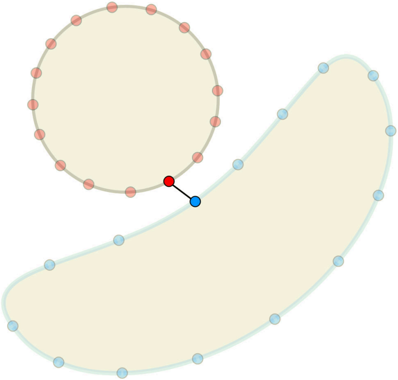

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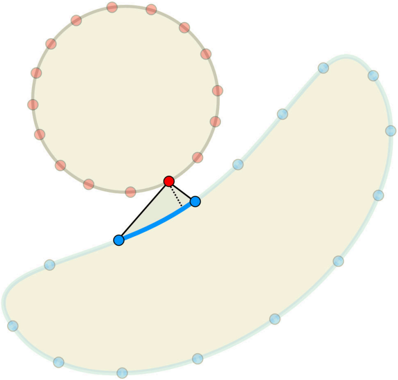
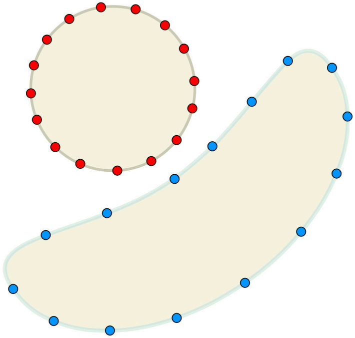


Acceleration of detection

- “Blind” algorithm ⇒ one-by-one



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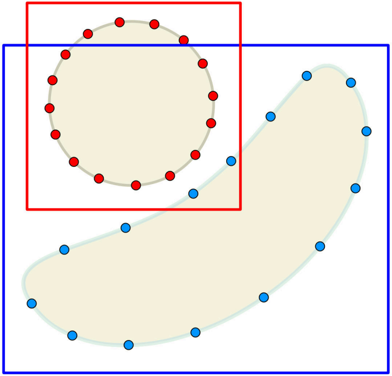

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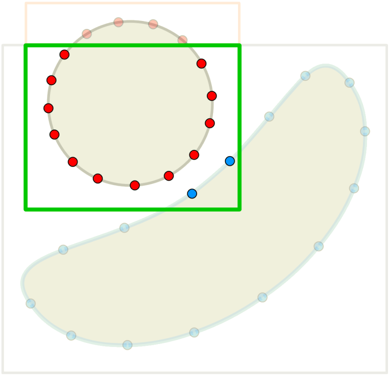

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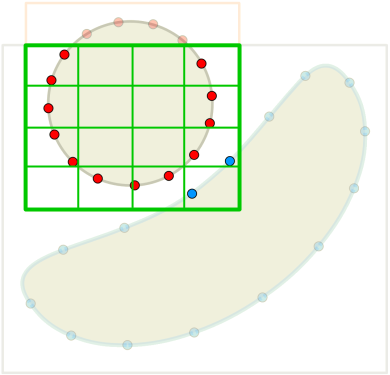

[1] Benson, Hallquist, 1991

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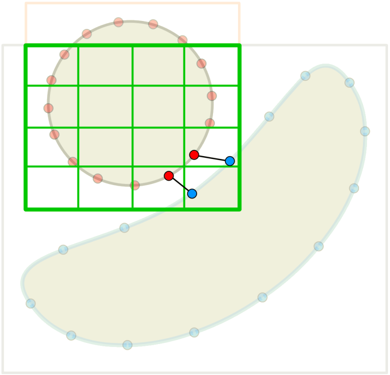

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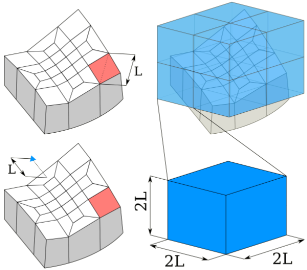
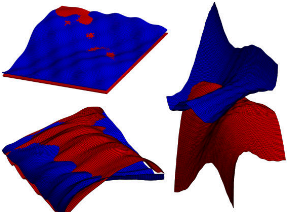
Acceleration of detection

- Strong connection between:
  - finite element mesh L,
  - maximal detection distance L,
  - bucket's size $2L$.
- User friendly algorithm
- Complexity $O(N)$



Relations between the master mesh, maximal detection distance and bucket's dimensions



Meshes for numerical tests

Unknown master-slave

- Master-slave may be unknown in advance:
  - complex geometry;
  - large sliding;
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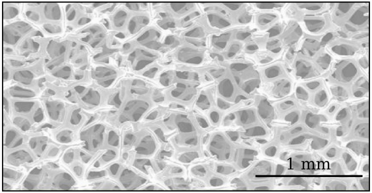
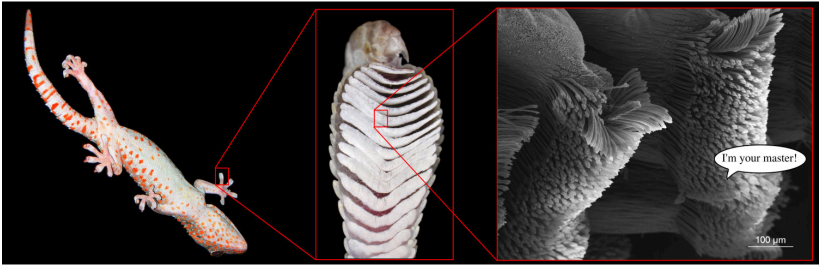


Nickel foam microstructure



Microstructure of gecko's adhesive toe
(adapted from Autumn Lab, Lewis& Clark Colledge, Portland, Oregon)

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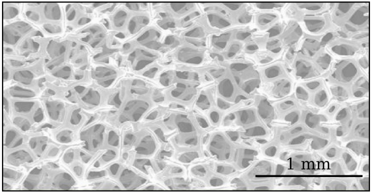

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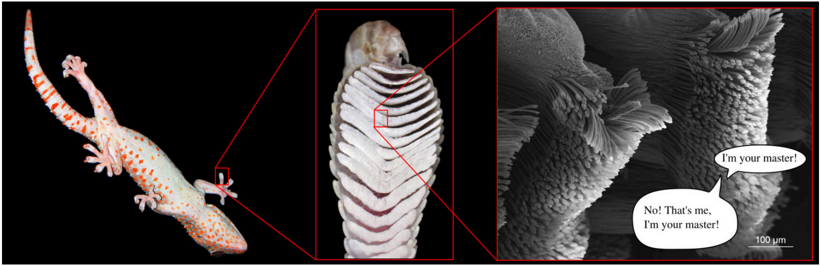

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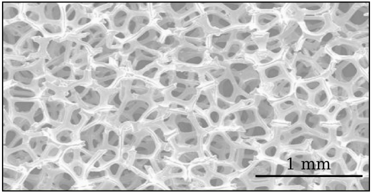

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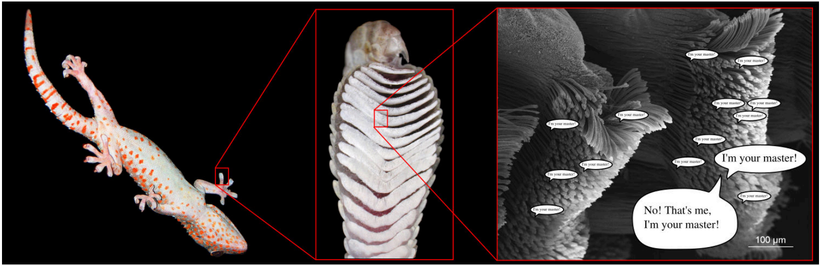

Microstructure of gecko's adhesive toe
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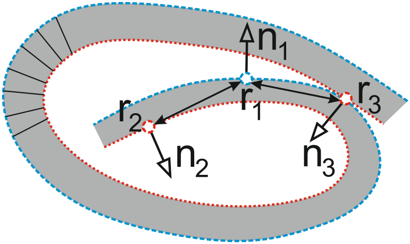
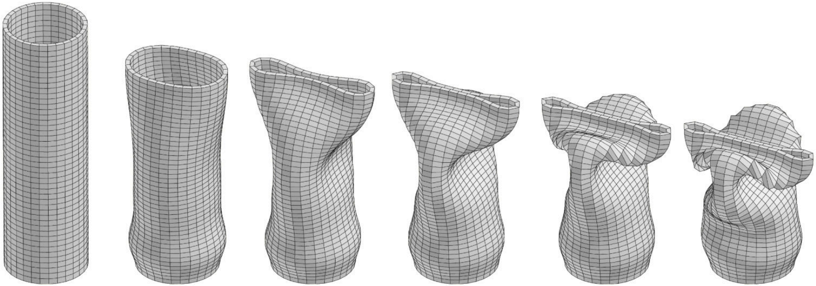
- Unknown master-slave
- The same algorithm
- Account of the nodal normals





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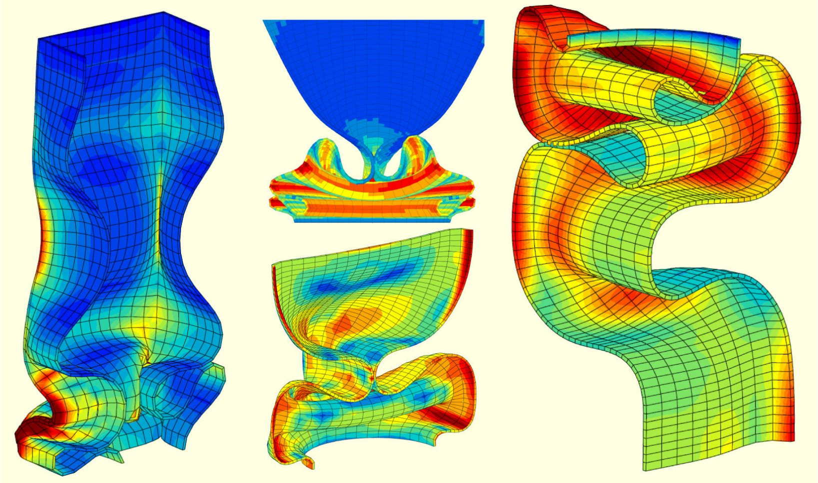
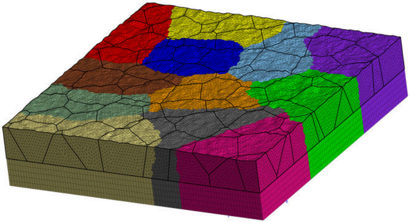
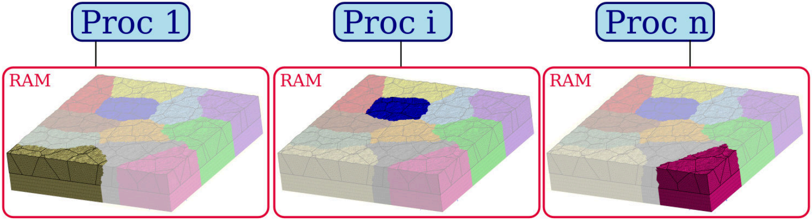


Finite Element Analyses of post-buckling behavior of thin-walled structures
(self-contact, finite strain plasticity) Zset/Zébulon

Parallelization

- Distributed memory computer architecture
- ⇒ Distributed contact surface
- No information about the entire contact surface





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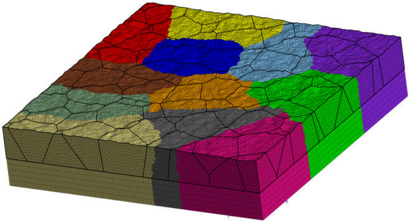
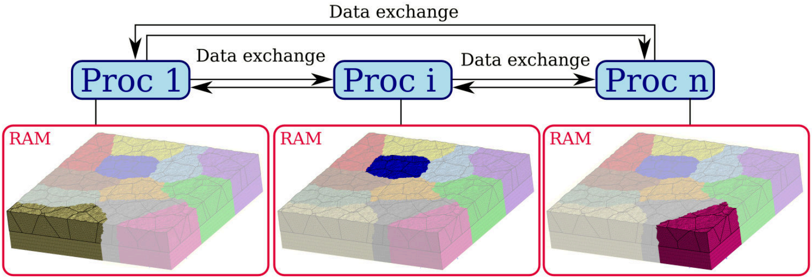
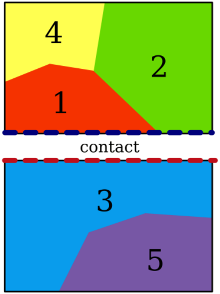
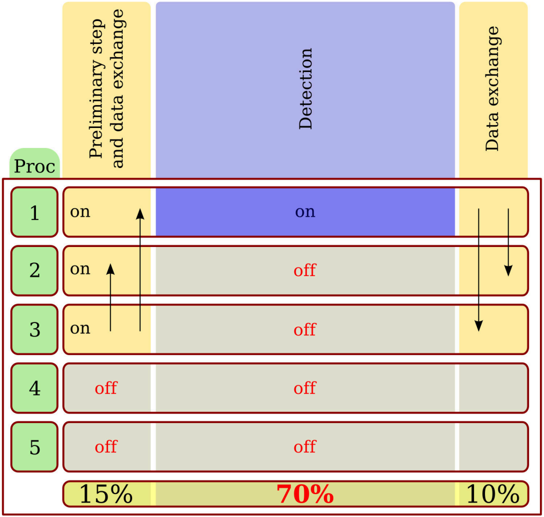




Detection by a single CPU

- SDMR
  Single Detection,
  Multiple Resolution
- Not optimal
- Simple data exchange





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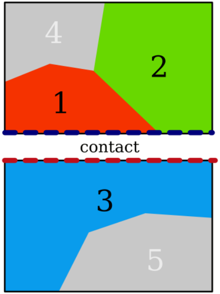
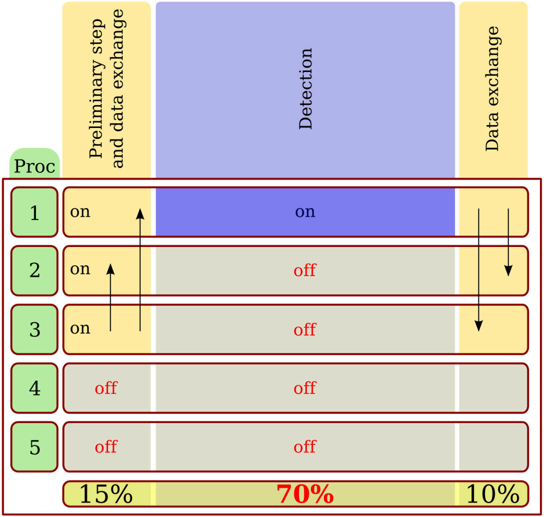
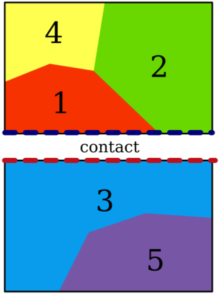
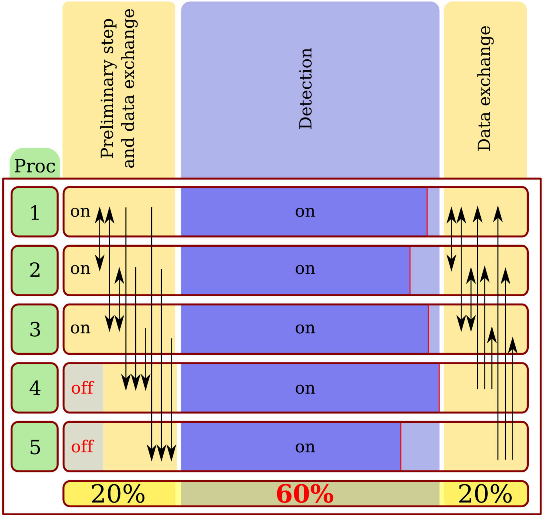




Distributed detection

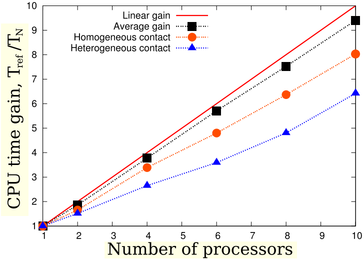
- MDMR
  Multiple Detection,
  Multiple Resolution
- More optimal
- Complex data exchange

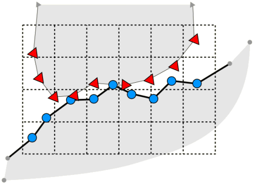




Distributed detection II

- Global detection bounding box
- Split into N equal overlapping parts
- One bucket overlap
- Test





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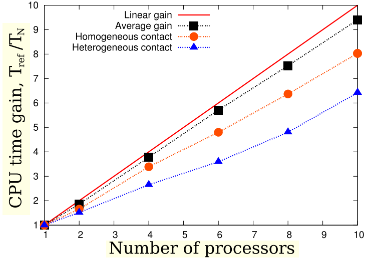

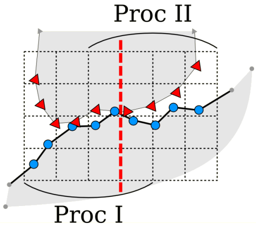

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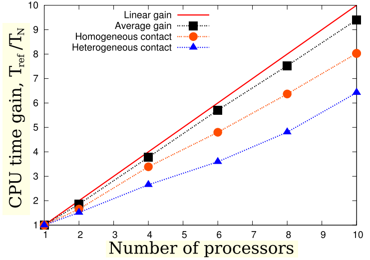

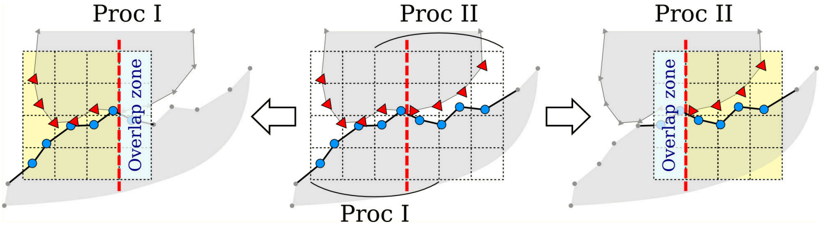

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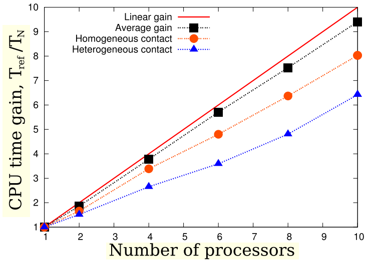
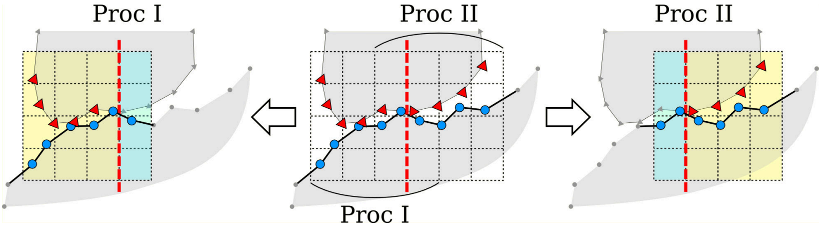
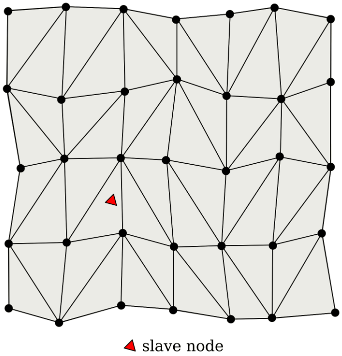




Methods based on the closest node detection are not robust

- Idea:
  - find the closest master node
  - find a projection on the adjacent segments
- Counterexample
- Closest segment is not always attached to the closest node
  - not regular mesh
  - triangular mesh
- Carefull use or improvement



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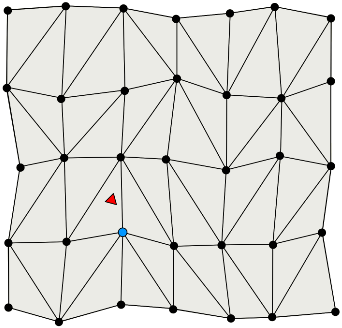

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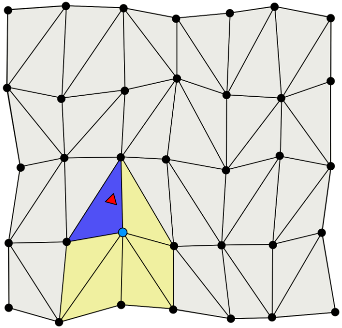

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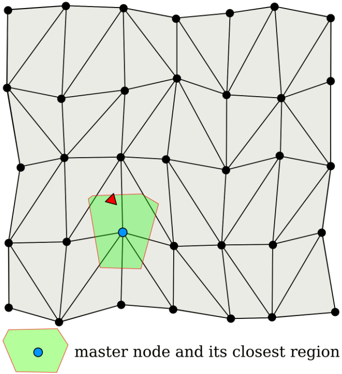

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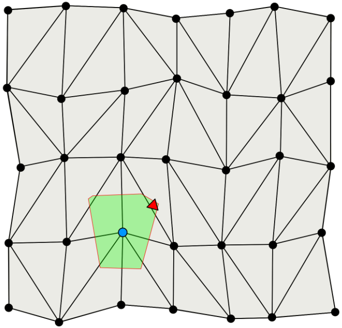

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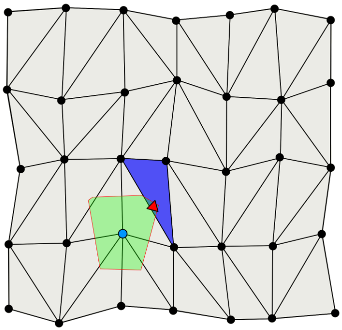

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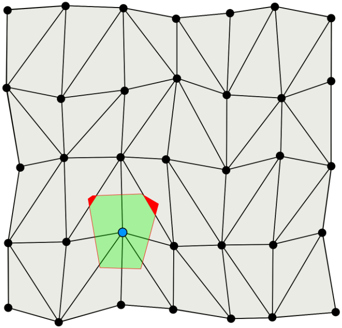

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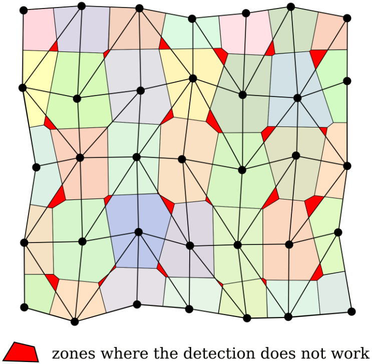

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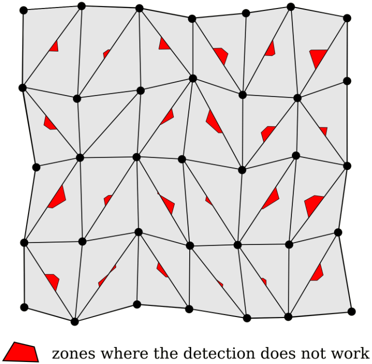

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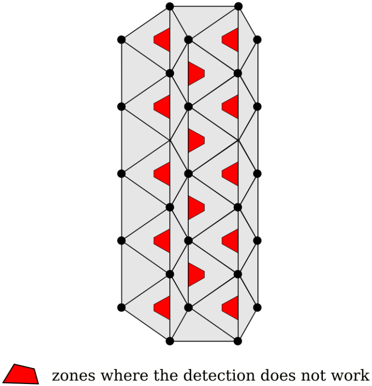
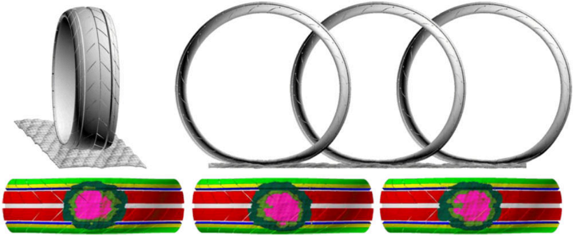
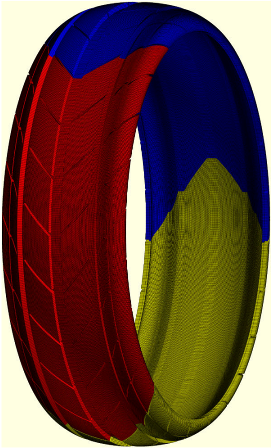


- Tyre-road problem
  - Tyre – 100 000 slave nodes
  - Road – 200 000 master segments
  - Detection **1.5-2** seconds



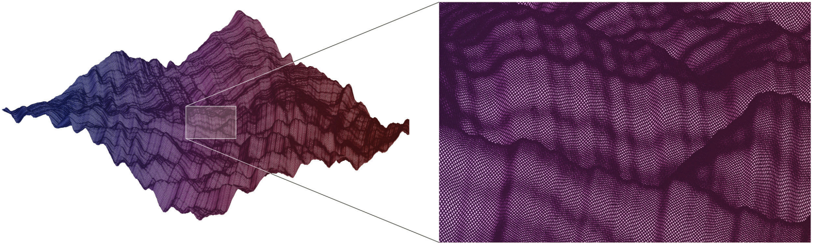
Contact elements for different loads
Zset/Zébulon



FE mesh of a tire
550 000 nodes, 105 000 slave nodes
Zset/Mesher

Example II

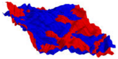
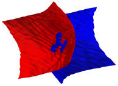
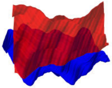
- Two curved surfaces in contact
  - 10^6 against 10^6 contact nodes
  - All-to-all $T_{\text{all-to-all}}$ **>180 hours**
  - Bucket sort performance depends on geometry:



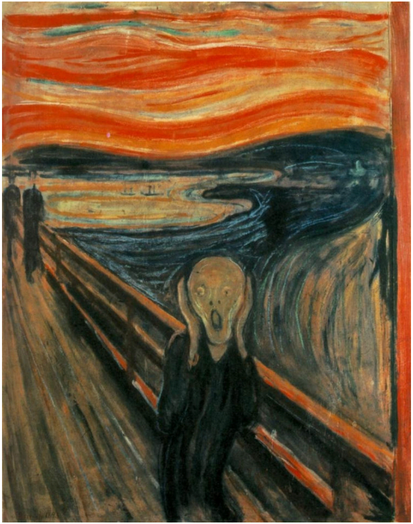
FE mesh of one of the contacting surfaces
Zset/Mesher

Example II

- Two curved surfaces in contact
  - 10^6 against 10^6 contact nodes
  - All-to-all $T_{\text{all-to-all}}$ **>180 hours**
  - Bucket sort performance depends on geometry:

Geometry	Nodes in bounding box	CPU time	Gain, $T_{\text{all-to-all}}/T_{\text{bucket}}$
	2 100 000	35 minutes	>300 times
	340 000	1 minute	>10 500 times
	50 000	4 seconds	>160 000 times

Friction . . . . . .



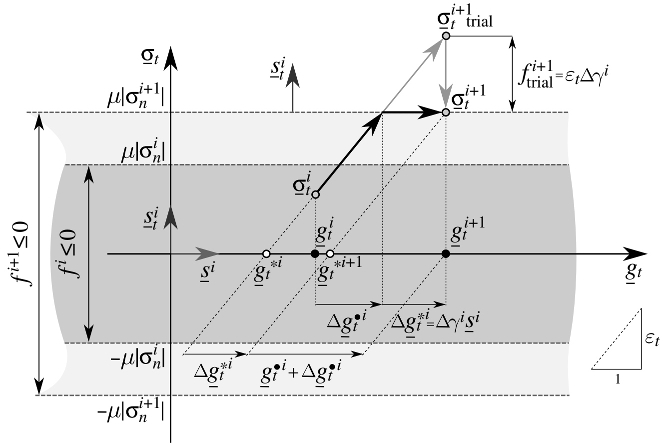
"The scream"

Friction: methods

- Optimization methods: penalty or augmented Lagrangian method
- Note that the method of Lagrange multipliers cannot be employed here
- Return mapping algorithm for penalty
- Analogy with elasto-plastic formulation problem[1]

[1] Curnier “A theory of friction” Int J Solids Struct 20 (1984)

- Return mapping algorithm in 2D for the penalty method

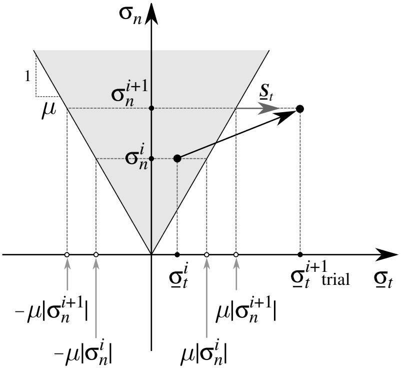


As in plasticity[1]

[1] Simo J.C. and Hughes T.J.. Computational inelasticity. Springer (2006)

Friction: Return mapping algorithm

- Return mapping algorithm in 2D for the penalty method



Analogy with non-associated plastic flow[2]

[2] Curnier A. A theory of friction. International Journal of Solids and Structures 20 (1984)

Application to contact problems: linearization

- Non-linear equation

$$R(\underline{u},\underline{f}) = 0$$

- Contains $\delta g_n, \delta \underline{g}_t$
- Use Newton-Raphson method
- Initial state at step i

$$R(\underline{u}^i,\underline{f}^i) = 0$$

- Should be also satisfied at step $i+1$

$$R(\underline{u}^{i+1},\underline{f}^{i+1}) = R(\underline{u}^i + \delta\underline{u},\underline{f}^{i+1}) = 0$$

- Linearize

$$R(\underline{u}^i + \delta\underline{u},\underline{f}^{i+1}) = R(\underline{u}^i,\underline{f}^{i+1}) + \frac{\partial R(\underline{u})}{\partial \underline{u}}\delta\underline{u} = 0$$

- Finally

$$\delta\underline{u} = -\underbrace{\left[\frac{\partial R(\underline{u})}{\partial \underline{u}}\right]^{-1}}_{\text{contains } \Delta\delta g_n, \Delta\delta \underline{g}_t} R(\underline{u}^i)$$

- NB: Contact problem does not satisfy conditions of Kantorovich theorem on the convergence of Newton's method.

Normal gap

- First variation enters in the residual vector:

$$\boxed{\delta g_n = \underline{\boldsymbol{n}} \cdot (\delta \underline{\boldsymbol{r}}_s - \delta \underline{\boldsymbol{\rho}})}$$

- Second variation enters in the tangent matrix:

$$\boxed{\begin{aligned}\Delta\delta g_n = &- \underline{\boldsymbol{n}} \cdot \left(\delta \frac{\partial \underline{\rho}}{\partial \underset{\sim}{\xi}}^T \Delta \underset{\sim}{\xi} + \Delta \frac{\partial \underline{\rho}}{\partial \underset{\sim}{\xi}}^T \delta \underset{\sim}{\xi} \right) - \Delta \underset{\sim}{\xi}^T \underset{\approx}{H} \delta \underset{\sim}{\xi} + \\ &+ g_n \left(\Delta \underset{\sim}{\xi}^T \underset{\approx}{H} + \underline{\boldsymbol{n}} \cdot \Delta \frac{\partial \underline{\rho}}{\partial \underset{\sim}{\xi}}^T \right) \underset{\approx}{\bar{A}} \left(\underline{\boldsymbol{n}} \cdot \delta \frac{\partial \underline{\rho}}{\partial \underset{\sim}{\xi}} + \underset{\approx}{H} \delta \underset{\sim}{\xi} \right)\end{aligned}}$$

Convective coordinate of the projection

- First variation enters in the residual vector:

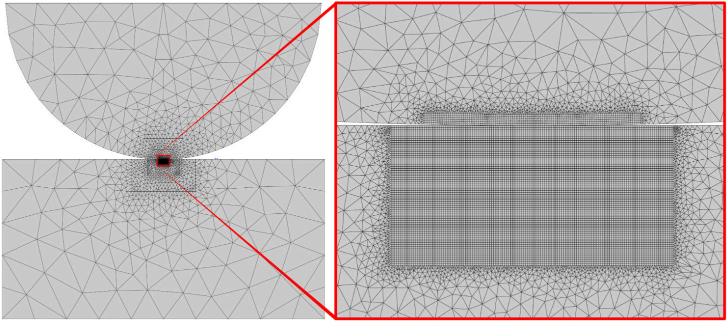
$$\boxed{\delta \underset{\sim}{\xi} = \left[\underset{\approx}{A} - g_n \underset{\approx}{H} \right]^{-1} \left(\frac{\partial \underline{\rho}}{\partial \underset{\sim}{\xi}} \cdot (\delta \underline{\boldsymbol{r}}_s - \delta \underline{\rho}) + g_n \underline{\boldsymbol{n}} \cdot \delta \frac{\partial \underline{\rho}}{\partial \underset{\sim}{\xi}} \right)}$$

- Second variation enters in the tangent matrix:

$$\boxed{\begin{aligned}
\Delta\delta \underset{\sim}{\xi} = (g_n \underset{\approx}{H} - \underset{\approx}{A})^{-1} \Bigg\{ & \frac{\partial \underline{\rho}}{\partial \underset{\sim}{\xi}} \cdot \left(\delta \frac{\partial \underline{\rho}}{\partial \underset{\sim}{\xi}}^{T} \Delta \underset{\sim}{\xi} + \Delta \frac{\partial \underline{\rho}}{\partial \underset{\sim}{\xi}}^{T} \delta \underset{\sim}{\xi} \right) + \Delta \underset{\sim}{\xi}^{T} \left(\frac{\partial \underline{\rho}}{\partial \underset{\sim}{\xi}} \cdot \frac{\partial^2 \underline{\rho}}{\partial \underset{\sim}{\xi}^2} \right) \delta \underset{\sim}{\xi} - \\
& - g_n \underline{\boldsymbol{n}} \cdot \left(\delta \frac{\partial^2 \underline{\rho}}{\partial \underset{\sim}{\xi}^2} \Delta \underset{\sim}{\xi} + \Delta \frac{\partial^2 \underline{\rho}}{\partial \underset{\sim}{\xi}^2} \delta \underset{\sim}{\xi} \right) - g_n \Delta \underset{\sim}{\xi}^{T} \left(\underline{\boldsymbol{n}} \cdot \frac{\partial^3 \underline{\rho}}{\partial \underset{\sim}{\xi}^3} \right) \delta \underset{\sim}{\xi} + \\
& + \left[g_n \left(\delta \frac{\partial \underline{\rho}}{\partial \underset{\sim}{\xi}} + \frac{\partial^2 \underline{\rho}}{\partial \underset{\sim}{\xi}^2} \delta \underset{\sim}{\xi} \right) \cdot \frac{\partial \underline{\rho}}{\partial \underset{\sim}{\xi}}^{T} \underset{\approx}{\bar{A}} - \delta g_n \underset{\approx}{I} \right] \left(\underline{\boldsymbol{n}} \cdot \Delta \frac{\partial \underline{\rho}}{\partial \underset{\sim}{\xi}} + \underset{\approx}{H} \Delta \underset{\sim}{\xi} \right) + \\
& + \left[g_n \left(\Delta \frac{\partial \underline{\rho}}{\partial \underset{\sim}{\xi}} + \frac{\partial^2 \underline{\rho}}{\partial \underset{\sim}{\xi}^2} \Delta \underset{\sim}{\xi} \right) \cdot \frac{\partial \underline{\rho}}{\partial \underset{\sim}{\xi}}^{T} \underset{\approx}{\bar{A}} - \Delta g_n \underset{\approx}{I} \right] \left(\underline{\boldsymbol{n}} \cdot \delta \frac{\partial \underline{\rho}}{\partial \underset{\sim}{\xi}} + \underset{\approx}{H} \delta \underset{\sim}{\xi} \right) \Bigg\}
\end{aligned}}$$

Particularities: mesh and convergence

- Strong **mesh refinement** is required
  - especially at **unknown edges** of contact zones



Typical mesh for fretting analysis [L. Sun, H. Proudhon, G. Cailletaud, 2011]

2D ∼ 30 000 DoFs,   3D ∼ 5 000 000 DoFs

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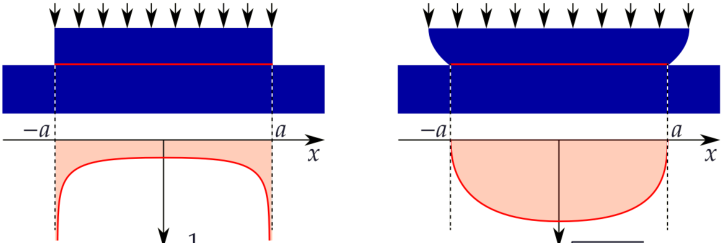


$$\sigma_n \sim \frac{1}{\sqrt{a^2 - x^2}} \qquad\qquad \sigma_n \sim \sqrt{a^2 - x^2}$$

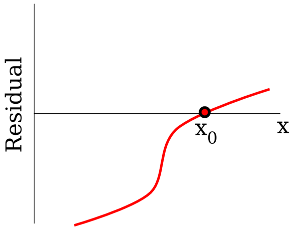
$$\sigma_n \xrightarrow[x\to a]{} -\infty \quad \left|\frac{\partial \sigma_n}{\partial x}\right| \xrightarrow[x\to a]{} \infty \qquad\qquad \left|\frac{\partial \sigma_n}{\partial x}\right| \xrightarrow[x\to a]{} \infty$$

Infinite contact pressure and/or its derivative

Particularities: mesh and convergence

- Strong **mesh refinement** is required
  - especially at **unknown edges** of contact zones
- **Slow change** of boundary conditions:
  - strong non-linearities of contact / friction problems
  - non-uniqueness of solution for frictional problems

Infinite looping



Initial guess $R(x_0, f_0) = 0$

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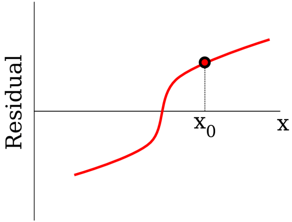

Too rapid change in boundary conditions $R(x_0, f_1) \neq 0$

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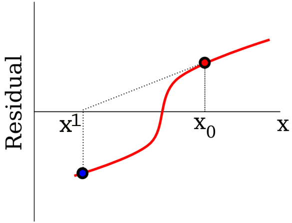

Iterations of Newton-Raphson method

$$R(x_0, f_1) + \left.\frac{\partial R}{\partial x}\right|_{x_0} \delta x = 0 \rightarrow \delta x = - \left.\frac{\partial R}{\partial x}\right|_{x_0}^{-1} R(x_0, f_1) \rightarrow x^1 = x_0 + \delta x$$

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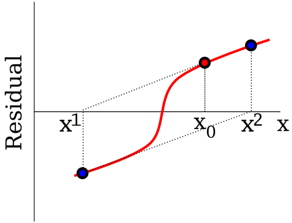

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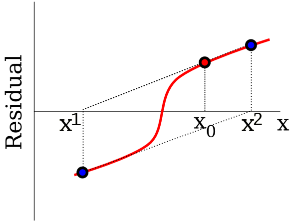

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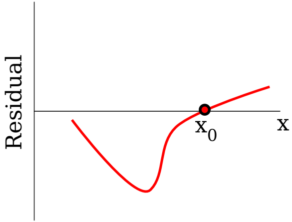
Convergence to a "false" solution



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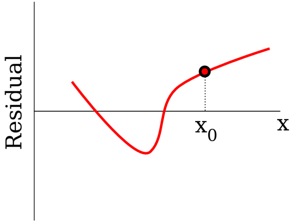

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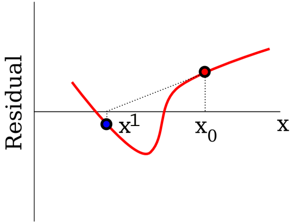

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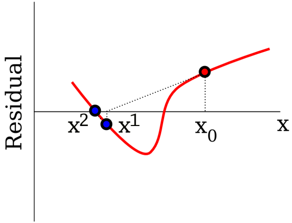

Iterations of Newton-Raphson method

$$R(x^1,f_1) + \frac{\partial R}{\partial x}\Big|_{x^1}\delta x = 0 \to \delta x = -\frac{\partial R}{\partial x}\Big|_{x^1}^{-1} R(x^1,f_1) \to x^2 = x^1 + \delta x$$

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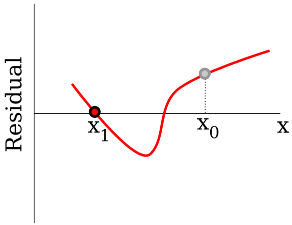
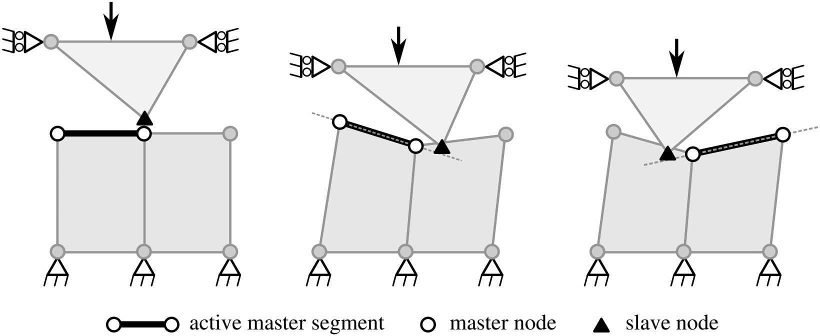


Convergence, but is it a "true" solution ?

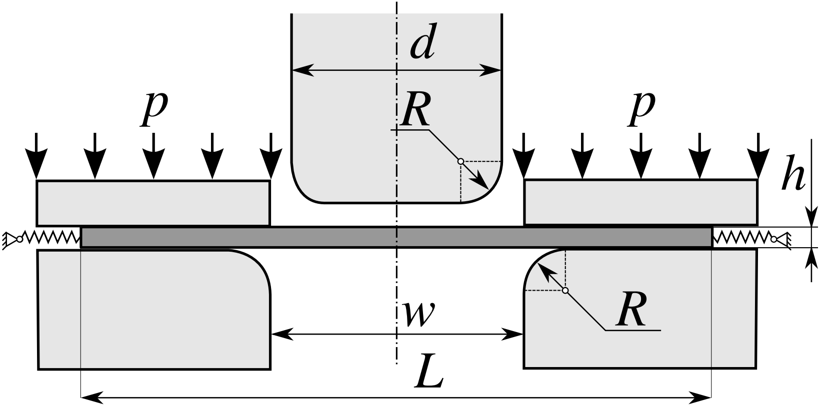
- **Infinite looping, e.g.**



- Change of the contact state (contact/non-contact, stick/slip)
- Interplay between stiffness, friction and augmented Lagrangian coefficients[1]
- Combination of non-linearities (e.g., plasticity+contact)

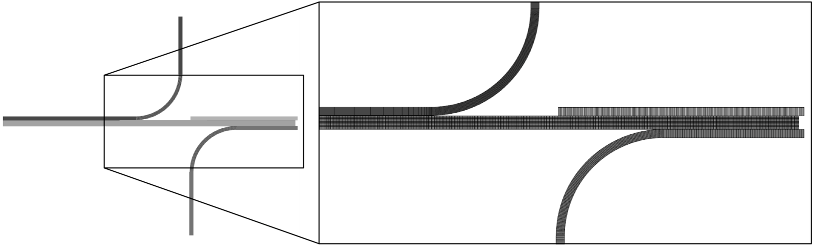
Alart P., Journal de Mathématiques Pures et Appliqués 76 (1997)

- Simulation of a deep drawing problem
- Dinite strain plasticity + frictional contact



Convergence problems: examples

- Simulation of a deep drawing problem
- Dinite strain plasticity + frictional contact



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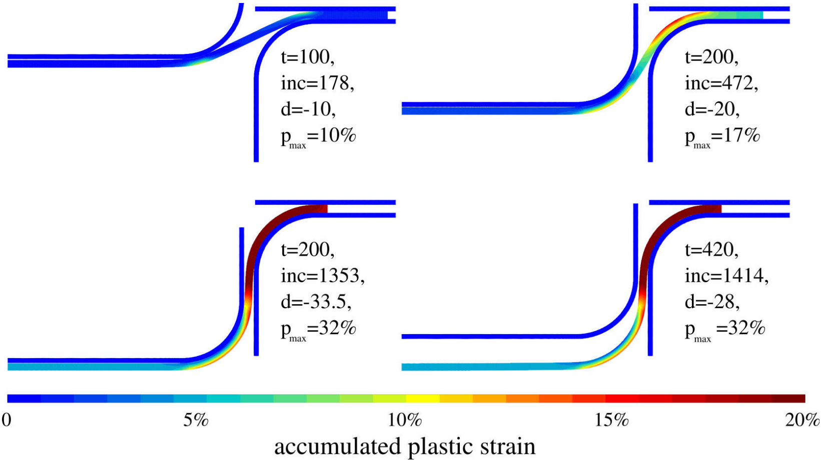

Convergence problems: examples

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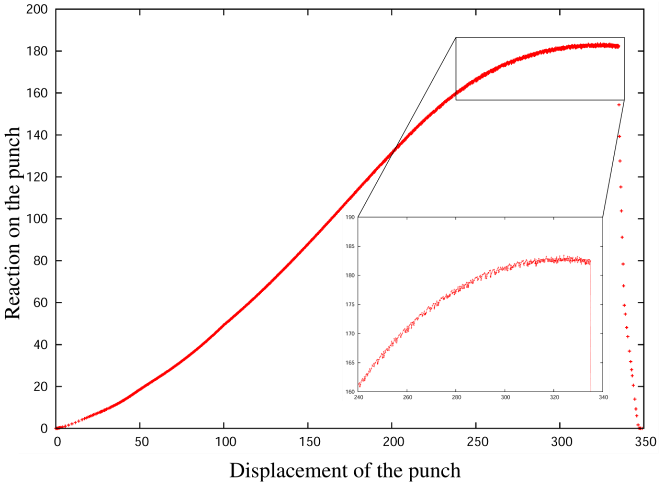
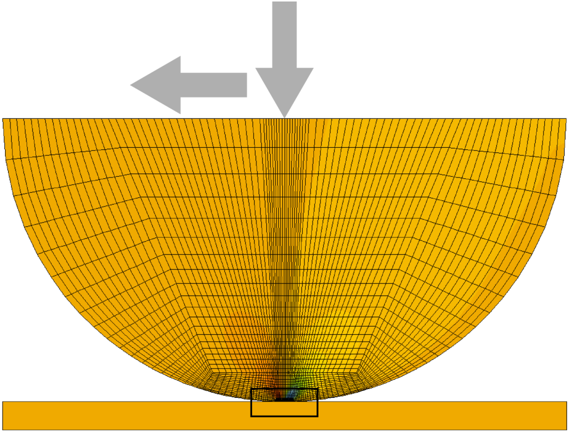


Cylinder-plane frictional contact

- Non-conservative problem, history of loading is crucial



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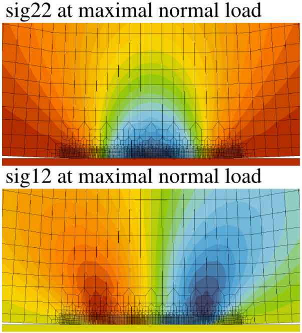
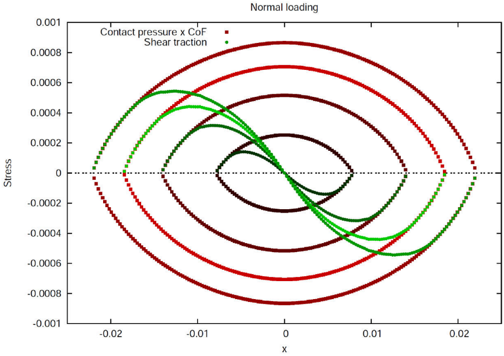


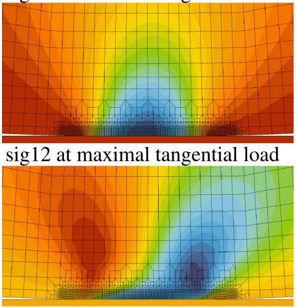


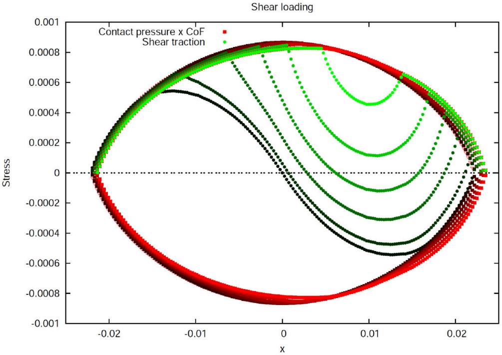
Press in 100 increments, $u_z \sim t^2$

- Non-conservative problem, history of loading is crucial

sig22 at maximal tangential load

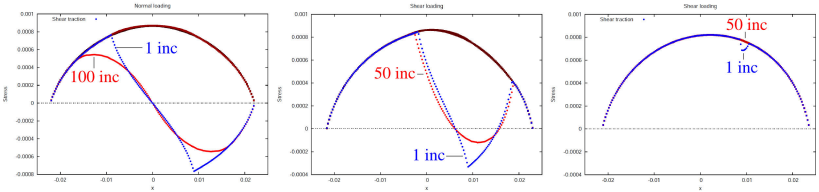
sig12 at maximal tangential load





Shift in 100 increments, $u_z \sim t$

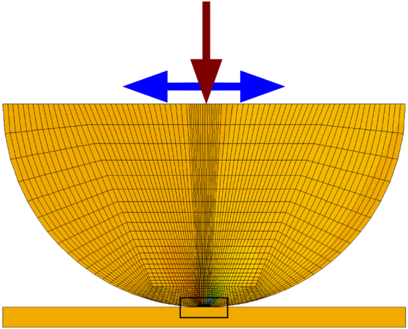
- Non-conservative problem, history of loading is crucial

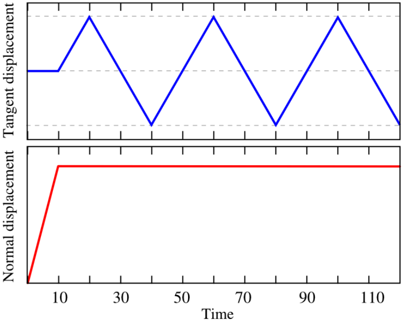


Comparison with: press in 1 increment, shift in 2 increments

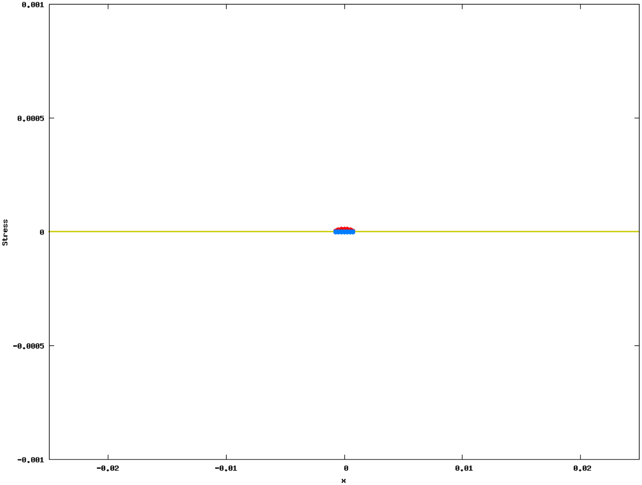
Before stick every point of the contact interface has to pass through the slip zone. It is impossible when loaded too fast.

Sphere-plane frictional contact: cycling

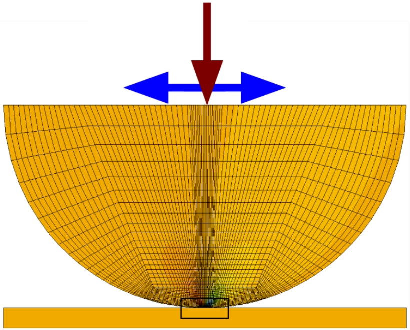


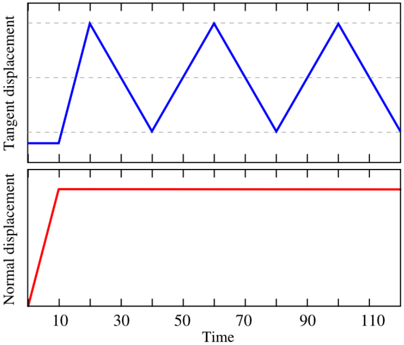


Sphere-plane frictional contact: cycling

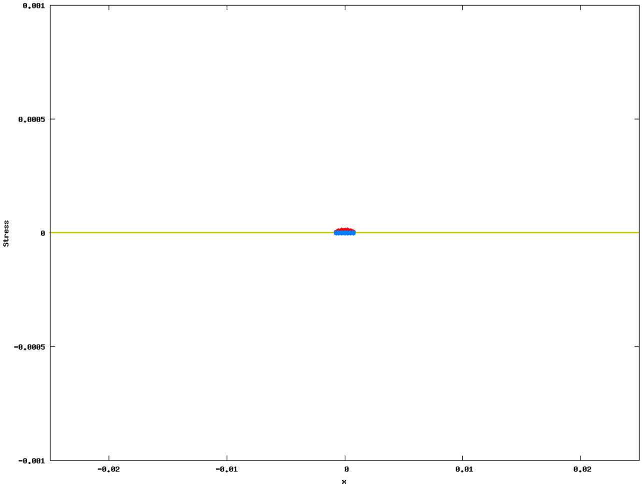


Sphere-plane frictional contact: cycling



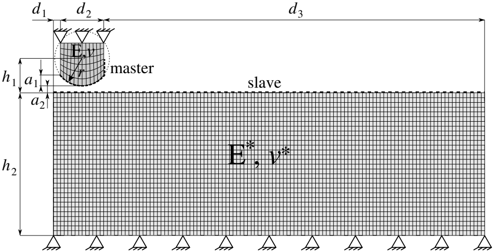


Sphere-plane frictional contact: cycling



Shallow ironing test

- Deformable-on-deformable frictional sliding
- Results obtained by different groups[1,2,3,4,5,6] differ significantly
- Local and global friction coefficients may differ



[1] Fischer K. A., Wriggers P., "Mortar based frictional contact formulation for higher order interpolations using the moving friction cone", Computer Methods in Applied Mechanics and Engineering, vol. 195, p. 5020-5036, 2006.

[2] Hartmann S., Oliver J., Cante J. C., Weyler R., Hernández J. A., "A contact domain method for large deformation frictional contact problems. Part 2: Numerical aspects", Computer Methods in Applied Mechanics and Engineering, vol. 198, p. 2607-2631, 2009.

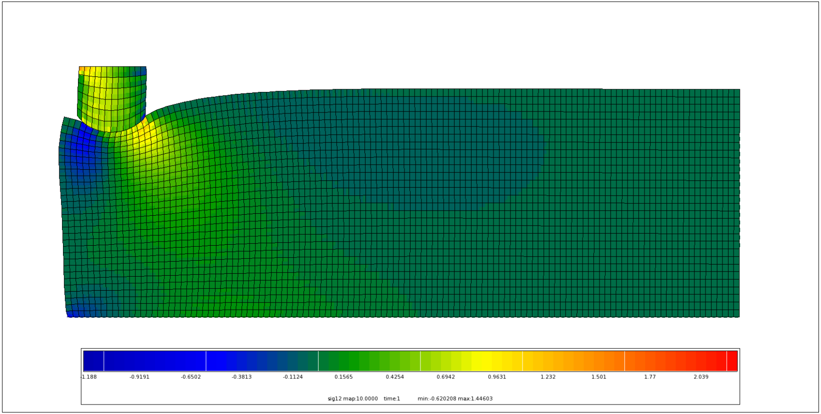
[3] Yastrebov V. A., "Computational contact mechanics: geometry, detection and numerical techniques", Thèse CdM & Onera, 2011.

[4] Kudawoo A. D., "Problèmes industriels de grande dimension en mécanique numérique du contact : performance, fiabilité et robustesse", Thse @ LMA & LAMSID, 2012.

[5] Poulios K., Renard Y., "A non-symmetric integral approximation of large sliding frictional contact problems of deformable bodies based on ray-tracing", soumis, 2014.

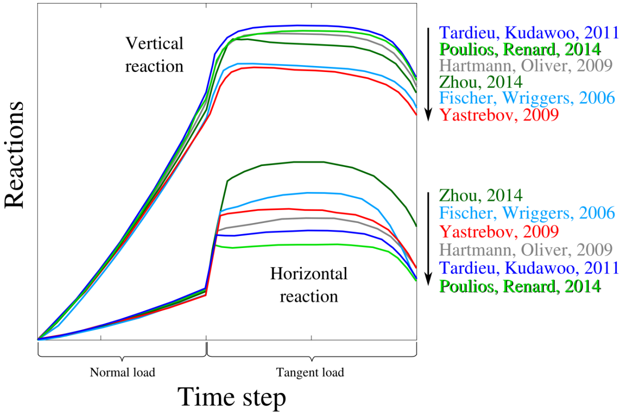
[6] Zhou Lei's blog, http://kt2008plus.blogspot.de

Shallow ironing test



Shallow ironing test

- No agreement between authors
- Dif. authors used dif. meshes (quadrilateral lin./sq., triangular lin.)
- Dif. authors used either finite or infinitesimal strain formulation



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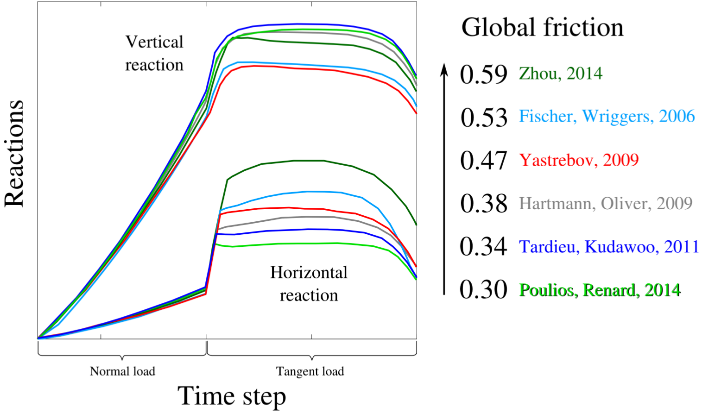


Local coefficient of friction $\mu_l = 0.3$

Examples of contact problems

With analytical solution

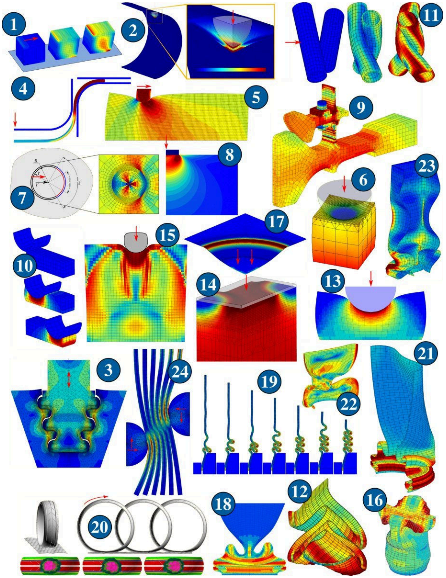
- ⋆ linear elasticity
- ⋆ with/without friction

From literature

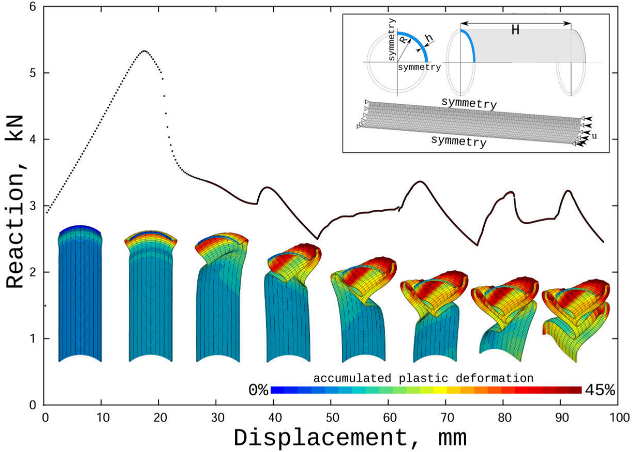
- ⋆ post-buckling 2D
- ⋆ finite strains
- ⋆ elasticity / plasticity
- ⋆ with/without friction

New

- ⋆ multi-contacts
- ⋆ post-buckling 3D
- ⋆ finite strains
- ⋆ elasticity / plasticity
- ⋆ with/without friction



Self-contact problem

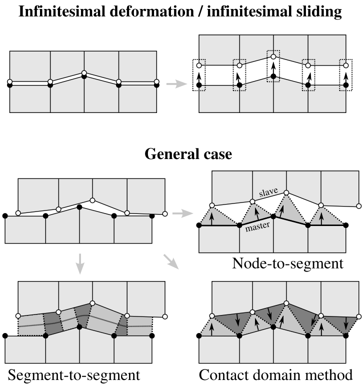


Finite element analysis of a post-buckling behavior of a thin walled tube

Collection of non-linearities: buckling instability, self-contact, finite strain plasticity

Reading

- It's just a tip of the "Computational Contact Mechanics" iceberg
- Contact detection
- Contact discretization and integration
- Smoothing techniques
- Energy conservative methods for dynamics

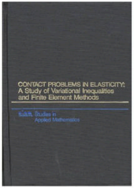


Contact discretization techniques

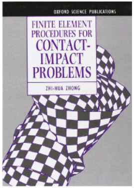
Reading

- It's just a tip of the "Computational Contact Mechanics" iceberg
- Contact detection
- Contact discretization and integration
- Smoothing techniques
- Energy conservative methods for dynamics

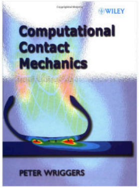
Kikuchi, Oden (1988)



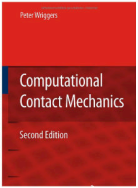
Zhong (1993)



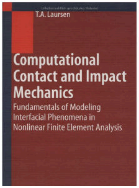
Wriggers (2002)



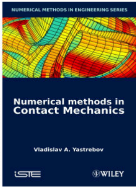
Wriggers, 2nd ed. (2006)



Laursen (2002)



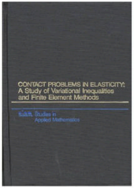
Yastrebov (2013)



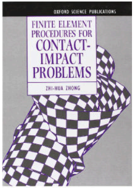
Reading

- It's just a tip of the "Computational Contact Mechanics" iceberg
- Contact detection
- Contact discretization and integration
- Smoothing techniques
- Energy conservative methods for dynamics
- Several advanced topics
  see `Yastrebov_CEMEF.pdf`, page 18.

Kikuchi, Oden (1988)



Zhong (1993)



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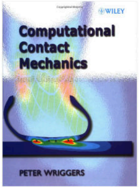

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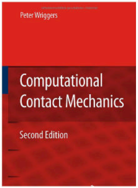

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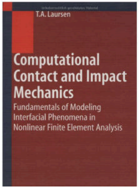

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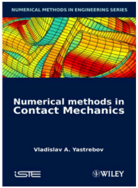

$\mathcal{L}_a(x, \lambda)$ Thank you for your attention!

In October 2 PhD positions open in
(computational) contact mechanics.

www.yastrebov.fr